A final remark: One can give an alternative definition of curvature simply by using metric properties of the manifold X itself, not properties of its embedding in $\mathbf{R}^N$. For two dimensions, this fact was realized by Gauss and is called the *theorem egregium*. Chern has given an "intrinsic" proof of the Gauss-Bonnet theorem, that is, one that completely avoids embedding (see [16]). For an accessible two-dimensional version of this proof, see [17]. (A "piecewise linear" form of the intrinsic Gauss-Bonnet theorem is derived below, in Exercise 11.)

EXERCISES

1. Let $f: \mathbf{R}^2 \longrightarrow \mathbf{R}$ be a smooth function, and let $S \subset \mathbf{R}^3$ be its graph. Prove that the volume form on S is just the form dA described in Section 4.†

2. If S is an oriented surface in $\mathbf{R}^3$ and (n_1, n_2, n_3) is its unit normal vector, prove that the volume form is

$$n_1\, dx_2 \wedge dx_3 + n_2\, dx_3 \wedge dx_1 + n_3\, dx_2 \wedge dx_3.$$

In particular, show that the volume form on the unit sphere is

$$x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2.$$

[HINT: Show that the 2-form just described is the 2-form defined by: $(v, w) \longrightarrow \frac{1}{2} \det \begin{pmatrix} v \\ w \\ n \end{pmatrix}$ for pairs of vectors v and w in $\mathbf{R}^3$.]

3. Let $A : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a rotation (i.e., an orthogonal linear mapping). Show that the map of S^{n-1} onto S^{n-1} induced by A preserves the volume form (that is, A^* applied to the volume form gives the volume form back again).

4. Let $C : [a, b] \longrightarrow \mathbf{R}^3$ be a parametrized curve in $\mathbf{R}^3$. Show that its "volume" (i.e., the integral over C of the volume form) is just the arc length:

$$\int_a^b \left| \frac{dC}{dt} \right| dt.$$

5. Let f be a smooth map of the interval $[a, b]$ into the positive real numbers. Let S be the surface obtained by rotating the graph of f around the

†Volume form is unfortunate terminology in two dimensions. "Area form" would be better.

x axis in $\mathbf{R}^3$. There is a classical formula which says that the sur area of S is equal to

$$\int_a^b 2\pi f\sqrt{1+(f')^2}\,dt.$$

Derive this formula by integrating the volume form over S. [HINT: U Exercise 2.]

6. Prove that for the $n-1$ sphere of radius r in $\mathbf{R}^n$, the Gaussian curvatur is everywhere $1/r^{n-1}$. [HINT: Show only that the derivative of the Gaus map is everywhere just $1/r$ times the identity.]

7. Compute the curvature of the hyperboloid $x^2+y^2-z^2=1$ at the point $(1, 0, 0)$. [HINT: The answer is -1.]

8. Let $f=f(x, y)$ be a smooth function on $\mathbf{R}^2$, and let $S \subset \mathbf{R}^3$ be its graph. Suppose that

$$f(0)=\frac{\partial f}{\partial x}(0)=\frac{\partial f}{\partial y}(0)=0.$$

Let κ_1 and κ_2 be the eigenvalues of the Hessian

$$\frac{1}{2}\begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}, & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x}, & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} \quad \text{at } 0.$$

Show that the curvature of S at $(0, 0, 0)$ is just the product $\kappa_1\kappa_2$. [HINT: It's enough simply to consider the case $f(x, y)=\kappa_1 x^2+\kappa_2 y^2$. Why?]

9. A surface $S \subset \mathbf{R}^3$ is called "ruled" if through every point of S there passes a straight line contained in S. Show that a ruled surface has Gaussian curvature less than or equal to zero everywhere.

10. Let $T=T_{a,b}$ be the standard torus consisting of points in $\mathbf{R}^3$ that are a distance a from the circle of radius b in the xy plane $(a<b)$. At what points is the curvature positive, at what points is it negative, and at what points is it zero?

11. Let $T_1, T_2, \ldots, T_k$ be a collection of closed triangles in $\mathbf{R}^3$. The set $S=T_1 \cup \cdots \cup T_k$ is called a polyhedral surface if the following statements are true.† [See part (a) of Figure 4-5.]

†We learned about this "intrinsic" form of the Gauss-Bonnet theorem for polyhedral surfaces from Dennis Sullivan.

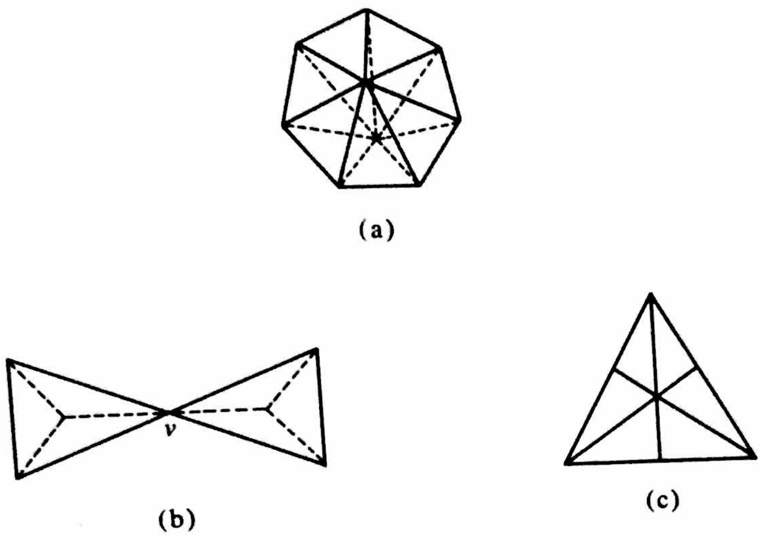

Figure 4-5

(a) Each side of T_i is also the side of exactly one other triangle T_j.
(b) No two triangles have more than one side in common.
(c) If $T_{i_1}, \ldots, T_{i_k}$ are the triangles in the collection having v as a vertex, and s_{i_k} is the side opposite v in T_{i_k}, then $\cup\, s_{i_k}$ is connected. [This disqualifies example (b) in Figure 4-5.]

For each vertex v we define $\kappa(v)$ to be 2π minus the sum of the angles at the vertex. Prove:

(i) If each triangle T_i is subdivided into smaller triangles (for example, as indicated in part (c) of Figure 4-5), the total sum $\sum \kappa(v)$ over all vertices is unchanged.

(ii)
$$\sum_v \kappa(v) = 2\pi \cdot \chi(S),$$

where the Euler characteristic $\chi(S)$ is defined to be number of vertices minus the number of sides plus the number of triangles. (Compare with Chapter 3, Section 7.) [HINT: Every triangle has three sides, and every side is contained in two triangles.]

12. Let X be an oriented $n - 1$ dimensional manifold and $f\colon X \to \mathbf{R}^n$ an immersion. Show that the Gauss map $g\colon X \to S^{n-1}$ is still defined even though X is not, properly speaking, a submanifold of $\mathbf{R}^n$. Prove that when X is even dimensional, the degree of g is one-half the Euler characteristic of X. Show this is not the case for odd-dimensional manifolds by showing that the Gauss map of the immersion

$$S^1 \longrightarrow \mathbf{R}^2, \qquad t \longrightarrow [\cos(nt), \sin(nt)]$$

has degree n.