$$\int_X f^*\omega = \int_X (h \circ f)^*\omega = \int_X f^*h^*\omega.$$

As $h^*\omega$ is supported in U, the lemma implies

$$\int_X f^*(h^*\omega) = \deg(f) \int_Y h^*\omega.$$

Finally, the diffeomorphism h is orientation preserving; for h ... implies $\deg(h) = +1$. Thus the change of variables property gives

$$\int_Y h^*\omega = \int_Y \omega,$$

and

$$\int_X f^*\omega = \deg(f) \int_Y \omega,$$

as claimed.

EXERCISES

1. Check that the 1-form $d \arg$ in $\mathbf{R}^2 - \{0\}$ is just the form

$$\frac{-y}{x^2 + y^2}\,dx + \frac{y}{x^2 + y^2}\,dy$$

discussed in earlier exercises. [HINT: $\theta = \arctan(y/x)$.] (This form is also often denoted $d\theta$.) In particular, you have already shown that d arg is closed but not exact.

2. Let γ be a smooth closed curve in $\mathbf{R}^2 - \{0\}$ and ω any closed 1-form on $\mathbf{R}^2 - \{0\}$. Prove that

$$\oint_\gamma \omega = W(\gamma, 0) \int_{S^1} \omega,$$

where $W(\gamma, 0)$ is the *winding number* of γ with respect to the origin. $W(\gamma, O)$ is defined just like $W_2(\gamma, 0)$, but using degree rather than degree mod 2; that is, $W(\gamma, O) = \deg(\gamma/|\gamma|)$. In particular, conclude that

$$W(\gamma, 0) = \frac{1}{2\pi} \oint_\gamma d\arg.$$

3. We can easily define *complex valued forms* on X. The forms we have used heretofore are *real forms*. Create imaginary p-forms by multiplying any real form by $i = \sqrt{-1}$. Then complex forms are sums $\omega_1 + i\omega_2$, where ω_1 and ω_2 are real. Wedge product extends in the obvious way, and d and $\int$ are defined to commute with multiplication by i:

$$d\omega = d\omega_1 + i\, d\omega_2, \qquad \int_X \omega = \int_X \omega_1 + i \int_X \omega_2.$$

Stokes theorem is valid for complex forms, for it is valid for their real and imaginary parts. We can now use our apparatus to prove a fundamental theorem in complex variable theory: the Cauchy Integral Formula.

(a) Let z be the standard complex coordinate function on $\mathbf{C} = \mathbf{R}^2$. Check that $dz = dx + i\, dy$.

(b) Let $f(z)$ be a complex valued function on an open subset U of $\mathbf{C}$. Prove that the 1-form $f(z)\, dz$ is closed if and only if $f(z) = f(x, y)$ satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}.$$

Express f in terms of its real and imaginary parts $f = f_1 + if_2$, and show that the Cauchy-Riemann equation is actually two real equations. If $f(z)\, dz$ is closed, the function f is called *holomorphic* in U.

(c) Show that the product of two holomorphic functions is holomorphic.

(d) Check that the complex coordinate function z is holomorphic. Conclude that every complex polynomial is holomorphic.

(e) Suppose that f is holomorphic in U and γ_1, γ_2 are two homotopic closed curves in U. Prove that

$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz.$$

[HINT: Use Exercise 9, Section 7.]

(f) If f is holomorphic in a simply connected region U, show that $\oint_{\gamma} f(z)\,dz = 0$ for every closed curve γ in U. [HINT: Part (e).]

(g) Prove that the function $f(z) = 1/z$ is holomorphic in the punctured plane $\mathbf{C} - \{0\}$. Similarly, $1/(z-a)$ is holomorphic in $\mathbf{C} - \{a\}$.

(h) Let C_r be a circle of radius r around the point $a \in \mathbf{C}$. Prove that

$$\int_{C_r} \frac{1}{z-a}\,dz = 2\pi i,$$

by direct calculation.

(i) Suppose that $f(z)$ is a holomorphic function in U and C_r is the circle of radius r around $a \in U$. Prove that

$$\int_{C_r} \frac{f(z)}{z-a}\,dz = 2\pi i \cdot f(a).$$

[HINT: By part (e), this does not depend on r. Note that $|f(z) - f(a)| < \epsilon_r$ on C_r, where $\epsilon_r \longrightarrow 0$ as $r \longrightarrow 0$. So let $r \longrightarrow 0$ and use a simple continuity argument.]

(j) Prove the *Cauchy Integral Formula*: If f is holomorphic in U and γ is a closed curve in U not passing through $a \in U$, then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a}\,dz = W(\gamma, a) \cdot f(a).$$

[HINT: Use part (i) and Exercise 2.]

4. Construct a k-form on S^k with nonzero integral. [HINT: Construct a compactly supported k-form in $\mathbf{R}^k$ with nonzero integral, and project stereographically.]

5. (a) Prove that a closed k-form ω on S^k is exact if and only if $\int_{S^k} \omega = 0$. [HINT: $\dim H^k(S^k) = 1$. Now use previous exercise.]

(b) Conclude that the linear map $\int_{S^k} : H^k(S^k) \longrightarrow \mathbf{R}$ is an isomorphism.

6. Prove that a compactly supported k-form ω on $\mathbf{R}^k$ is the exterior derivative of a compactly supported $k - 1$ form if and only if $\int_{\mathbf{R}^k} \omega = 0$. [HINT: Use stereographic projection to carry ω to a form ω' on S^k. By Exercise 5, $\omega' = dv$. Now dv is zero in a contractible neighborhood U of the north pole N. Use this to find a $k - 2$ form μ on S^k such that $v = d\mu$ near N. Then $v - d\mu$ is zero near N, so it pulls back to a compactly supported form on $\mathbf{R}^k$.]

7. Show that on any compact oriented k-dimensional manifold X, the linear map $\int_X : H^k(X) \longrightarrow \mathbf{R}$ is an isomorphism. In particular, show $\dim H^k(X) = 1$. [HINT: Let U be an open set diffeomorphic to $\mathbf{R}^k$, and let ω be a k-form compactly supported in U with $\int_X \omega = 1$. Use Exercise 6 to show that every compactly supported form in U is cohomologous to a scalar multiple of ω. Now choose open sets $U_1, \ldots, U_N$ covering X, each of which is deformable into U by a smooth isotopy. Use Exercise 7 of Section 6 and a partition of unity to show that any k-form θ on X is cohomologous to $c\omega$ for some $c \in \mathbf{R}$. Indentify c.]

8. Let $f: X \longrightarrow Y$ be a smooth map of compact oriented k-manifolds. Consider the induced map on the top cohomology groups, $f^{\#}: H^k(Y) \longrightarrow H^k(X)$. Integration provides canonical isomorphisms of both $H^k(Y)$ and $H^k(X)$ with $\mathbf{R}$, so under these isomorphisms the linear map $f^{\#}$ must correspond to multiplication by some scalar. Prove that this scalar is the degree of f. In other words, the following square commutes:

$$\begin{array}{ccc} H^k(Y) & \xrightarrow{\quad f^{\#} \quad} & H^k(X) \\ \Big\downarrow \int_Y & & \Big\downarrow \int_X \\ \mathbf{R} & \xrightarrow[\deg(f)]{\text{multiplication by}} & \mathbf{R} \end{array}$$

§9 The Gauss-Bonnet Theorem

We begin this section with a discussion of volume. Suppose that X is a compact oriented k-dimensional manifold in $\mathbf{R}^N$. For each point $x \in X$, let $v_X(x)$ be the volume element on $T_x(X)$, the alternating k-tensor that has value $1/k!$ on each positively oriented orthonormal basis for $T_x(X)$. (See Exercise 10, Section2.) It is not hard to show that the k-form v_X on X is smooth; it is called the *volume form* of X. For example, the volume form on