## EXERCISES

1. Show that Stokes theorem for a closed interval $[a, b]$ in $\mathbf{R}^1$ is just the Fundamental Theorem of Calculus. (See Exercise 1, Section 4.)

2. Prove the classical Green's formula in the plane: let $W$ be a compact domain in $\mathbf{R}^2$ with smooth boundary $\partial W = \gamma$. Prove

$$\int_\gamma f\,dx + g\,dy = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.$$

3. Prove the Divergence Theorem: let $W$ be a compact domain in $\mathbf{R}^3$ with smooth boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field on $W$. Then

$$\int_W (\operatorname{div} \vec{F})\,dx\,dy\,dz = \int_{\partial W} (\vec{n} \cdot \vec{F})\,dA.$$

(Here $\vec{n}$ is the outward normal to $\partial W$. See Exercises 13 and 14 of Section 4 for $dA$, and page 178 for div $\vec{F}$.)

4. Prove the classical Stokes theorem: let $S$ be a compact oriented two-manifold in $\mathbf{R}^3$ with boundary, and let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field in a neighborhood of $S$. Prove

$$\int_S (\operatorname{curl} \vec{F} \cdot \vec{n})\,dA = \int_{\partial S} f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3.$$

(Here $\vec{n}$ is the outward normal to $S$. For $dA$, see Exercises 13 and 14 of Section 4, and for curl $\vec{F}$, see page 178.)

**5.** The Divergence Theorem has an interesting interpretation in fluid dynamics. Let $D$ be a compact domain in $\mathbf{R}^3$ with a smooth boundary $S = \partial D$. We assume that $D$ is filled with an incompressible fluid whose density at $x$ is $\rho(x)$ and whose velocity is $\vec{v}(x)$. The quantity

$$\int_S \rho(\vec{v} \cdot \vec{n})\, dA$$

measures the amount of fluid flowing out of $S$ at any fixed time. If $x \in D$ and $B_\epsilon$ is the ball of radius $\epsilon$ about $x$, the "infinitesimal amount" of fluid being added to $D$ at $x$ at any fixed time is

$$(*) \quad \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \frac{\rho(\vec{v} \cdot \vec{n})\, dA}{\operatorname{vol}(B_\epsilon)}.$$

Show that $(*) = \operatorname{div} \rho\vec{v}$, and deduce from the Divergence Theorem that the amount of fluid flowing out of $D$ at any fixed time equals the amount being added.

**6.** The Divergence Theorem is also useful in electrostatics. Let $D$ be a compact region in $\mathbf{R}^3$ with a smooth boundary $S$. Assume $0 \in \operatorname{Int}(D)$. If an electric charge of magnitude $q$ is placed at 0, the resulting force field is $q\vec{r}/r^3$, where $\vec{r}(x)$ is the vector to a point $x$ from 0 and $r(x)$ is its magnitude. Show that the amount of charge $q$ can be determined from the force on the boundary by proving Gauss's law:

$$\int_S \vec{F} \cdot \vec{n}\, dA = 4\pi q.$$

[HINT: Apply the Divergence Theorem to a region consisting of $D$ minus a small ball around the origin.]

***7.** Let $X$ be compact and boundaryless, and let $\omega$ be an exact $k$-form on $X$, where $k = \dim X$. Prove that $\int_X \omega = 0$. [HINT: Apply Stokes theorem. Remember that $X$ is a manifold with boundary, even though $\partial X$ is empty.]

***8.** Suppose that $X = \partial W$, $W$ is compact, and $f: X \to Y$ is a smooth map. Let $\omega$ be a closed $k$-form on $Y$, where $k = \dim X$. Prove that if $f$ extends to all of $W$, then $\int_X f^*\omega = 0$.

***9.** Suppose that $f_0, f_1: X \to Y$ are homotopic maps and that the compact, boundaryless manifold $X$ has dimension $k$. Prove that for all closed $k$-

forms $\omega$ on $Y$,

$$\int_X f_0^*\omega = \int_X f_1^*\omega.$$

[HINT: Previous exercise.]

10. Show that if $X$ is a simply connected manifold, then $\oint_\gamma \omega = 0$ for all closed 1-forms $\omega$ on $X$ and all closed curves $\gamma$ in $X$. [HINT: Previous exercise.]

11. Prove that if $X$ is simply connected, then all closed 1-forms $\omega$ on $X$ are exact. (See Exercise 11, Section 4.)

12. Conclude from Exercise 11 that $H^1(S^k) = 0$ if $k > 1$.

13. Suppose that $Z_0$ and $Z_1$ are compact, cobordant, $p$-dimensional submanifolds of $X$. Prove that

$$\int_{Z_0} \omega = \int_{Z_1} \omega$$

for every closed $p$-form $\omega$ on $X$.

14. (a) Suppose that $\omega_1$ and $\omega_2$ are cohomologous $p$-forms on $X$, and $Z$ is a compact oriented $p$-dimensional submanifold. Prove that

$$\int_Z \omega_1 = \int_Z \omega_2.$$

(b) Conclude that integration over $Z$ defines a map of the cohomology group $H^p(X)$ into $\mathbf{R}$, which we denote by

$$\int_Z : H^p(X) \rightarrow \mathbf{R}.$$

Check that $\int_Z$ is a linear functional on $H^p(X)$.

(c) Suppose that $Z$ bounds; specifically, assume that $Z$ is the boundary of some compact oriented $p+1$ dimensional submanifold-with-boundary in $X$. Show that $\int_Z$ is zero on $H^p(X)$.

(d) Show that if the two compact oriented manifolds $Z_1$ and $Z_2$ in $X$ are cobordant, then the two linear functionals $\int_{Z_1}$ and $\int_{Z_2}$ are equal.