cohomology classes on X; that is, f^* defines a mapping $f^\#: H^p(Y) \longrightarrow H^p(X)$. Since f^* is linear, you can easily check that $f^\#$ is linear. (Remember that $f^\#$ *pulls back:* i.e., when $f: X \longrightarrow Y$, then $f^\#: H^p(Y) \longrightarrow H^p(X)$.)

The numbered statements in the following discussion are for you to prove.

1. If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)^\# = f^\# \circ g^\#$.

In some simple cases, we can easily compute $H^p(X)$. For example, $H^p(X) = 0$ for all $p >$ dim X. The next easiest case is

2. The dimension of $H^0(X)$ equals the number of connected components in X. [HINT: There are no exact zero forms. Show that a zero form—that is, a function—is closed if and only if it is constant on each component of X.]

In obtaining information about other cohomology groups, we shall define an operator P on forms. Just like the operators d and $\int$, P is first defined

$$P\omega(t, x) = \sum_I \left[\int_0^t f_I(s, x)\, ds\right] dx_I.$$

Notice that $P\omega$ does not involve a dt term.

Now let $\phi: V \longrightarrow U$ be a diffeomorphism of open subsets of $\mathbf{R}^k$, and let $\Phi: \mathbf{R} \times V \longrightarrow \mathbf{R} \times U$ be the diffeomorphism $\Phi = \text{identity} \times \phi$. Prove the essential transformation property

3. $\Phi^* P\omega = P\Phi^*\omega$. [Hint: This is not difficult if you avoid writing everything in coordinates. Just note that $\Phi^*\, dt = dt$ and that Φ^* converts each of the two sums in the expression (1) for ω into sums of the same type.]

Now copy the arguments used to put d and $\int$ on manifolds to prove

4. There exists a unique operator P, defined for all manifolds X, that transforms p-forms on $\mathbf{R} \times X$ into $p - 1$ forms on $\mathbf{R} \times X$ and that satisfies the following two requirements:

   (1) If $\phi: X \longrightarrow Y$ is a diffeomorphism, and $\Phi = \text{identity} \times \phi$, then $\Phi^* \circ P = P \circ \Phi^*$.

   (2) If X is an open subset of $\mathbf{R}^k$, P is as defined above.

The main attraction of this operator is the following marvelous formula. (No doubt it appears anything but marvelous at first, but wait!)

5. Let $\pi: \mathbf{R} \times X \longrightarrow X$ be the usual projection operator and $i_a: X \longrightarrow \mathbf{R} \times X$ be any embedding $x \longrightarrow (a, x)$. Then

$$dP\omega + Pd\omega = \omega - \pi^* i_a^* \omega.$$

[Hint: Essentially this is a question of unraveling notation. For example, if ω is expressed by the sum (1), then $\pi^* i_a^* \omega = \sum_J g_J(x, a)\, dx_J$.]

Say $\omega = g(x,t)$

The first important consequence

6. (*Poincaré Lemma*) The maps

$$i_a^{\#}: H^p(\mathbf{R} \times X) \longrightarrow H^p(X) \quad \text{and} \quad \pi^{\#}: H^p(X) \longrightarrow H^p(\mathbf{R} \times X)$$

are inverses of each other. In particular, $H^p(\mathbf{R} \times X)$ is isomorphic to $H^p(X)$.
[HINT: $\pi \circ i_a = \text{identity}$, so Exercise 1 implies $i_a^{\#} \circ \pi^{\#} = \text{identity}$. For $\pi^{\#} \circ i_a^{\#}$, interpret Exercise 5 for closed forms ω.]

Take X to be a single point, so $H^p(X) = 0$ if $p > 0$. Now the Poincaré lemma implies by induction:

Corollary. $H^p(\mathbf{R}^k) = 0$ if $k > 0$; that is, every closed p-form on R^k is exact if $p > 0$.

A slightly more subtle consequence is

7. If $f, g: X \longrightarrow Y$ are homotopic maps, then $f^{\#} = g^{\#}$. [HINT: Let $H: \mathbf{R} \times X \longrightarrow Y$ be a smooth map such that $H(a, x) = f(x)$ and $H(b, x) = g(x)$. Then

$$f^{\#} = i_a^{\#} \circ H^{\#} \quad \text{and} \quad g^{\#} = i_b^{\#} \circ H^{\#}.$$

But it is clear from Exercise 6 that $i_a^{\#} = i_b^{\#}$.]

Now strengthen the corollary to Exercise 6 by proving

8. If X is contractible, then $H^p(X) = 0$ for all $p > 0$.

We conclude this section with one deeper result.

Theorem. $H^p(S^k)$ is one dimensional for $p = 0$ and $p = k$. For all other p, $H^p(S^k) = 0$ $(k > 0)$.

the theorem for