$$= (\operatorname{div} \vec{F})\, dx_1 \wedge dx_2 \wedge dx_3.$$

3-forms. $d(\text{any 3-form}) = 0$. (Why?)

So the classical operators of vector calculus in 3-space—the gradient, curl, and divergence—are really the d operator in vector field form. Show that the cocycle condition $d^2 = 0$ on R^3 is equivalent to the two famous formulas curl (grad f) $= 0$ and div (curl $\vec{F}$) $= 0$.

EXERCISES

1. Calculate the exterior derivatives of the following forms in $\mathbf{R}^3$:
   (a) $z^2\, dx \wedge dy + (z^2 + 2y)\, dx \wedge dz$.
   (b) $13x\, dx + y^2\, dy + xyz\, dz$.
   (c) $f\, dg$, where f and g are functions.
   (d) $(x + 2y^3)(dz \wedge dx + \frac{1}{2} dy \wedge dx)$.

2. Show that the vector field

$$\left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$$

has curl zero, but that it cannot be written as the gradient of any function.

§6 Cohomology with Forms†

A p-form ω on X is *closed* if $d\omega = 0$ and *exact* if $\omega = d\theta$ for some $(p - 1)$ form θ on X. Exact forms are all closed, since $d^2 = 0$, but it may not be true that closed forms are all exact. In fact, whether or not closed forms on X are actually exact turns out to be a purely topological matter. ... expressed