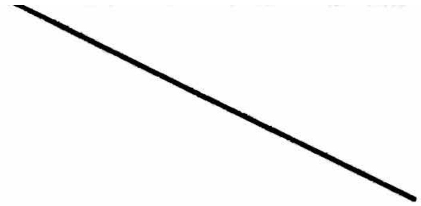

Figure 4-2

EXERCISES

1. Let Z be a finite set of points in X, considered as a 0-manifold. Fix an orientation of Z, an assignment of orientation numbers $\sigma(z) = \pm 1$ to each $z \in Z$. Let f be any function on X, considered as a 0-form, and check that

$$\int_Z f = \sum_{z \in Z} \sigma(z) \cdot f(z).$$

2. Let X be an oriented k-dimensional manifold with boundary, and ω a compactly supported k-form on X. Recall that $-X$ designates the oriented manifold obtained simply by reversing the orientation on X.

Check that

$$\int_{-X} \omega = -\int_X \omega.$$

3. Let $c: [a, b] \longrightarrow X$ be a smooth curve, and let $c(a) = p$, $c(b) = q$. Show that if ω is the differential of a function on X, $\omega = df$, then

$$\int_a^b c^*\omega = f(q) - f(p).$$

4. Let $c: [a, b] \longrightarrow X$ be a smooth curve, and let $f: [a_1, b_1] \longrightarrow [a, b]$ be a smooth map with $f(a_1) = a$ and $f(b_1) = b$. Show that the integrals

$$\int_a^b c^*\omega \quad \text{and} \quad \int_{a_1}^{b_1} (c \circ f)^*\omega$$

are the same (i.e., $\int_c \omega$ is independent of orientation-preserving reparametrization of c).

*5. A *closed curve* on a manifold X is a smooth map $\gamma: S^1 \longrightarrow X$. If ω is a 1-form on X, define the *line integral* of ω around γ by

$$\oint_\gamma \omega = \int_{S^1} \gamma^*(\omega).$$

For the case $X = \mathbf{R}^k$, write $\oint_\gamma \omega$ explicitly in terms of the coordinate expressions of γ and ω.

6. Let $h: \mathbf{R}^1 \longrightarrow S^1$ be $h(t) = (\cos t, \sin t)$. Show that if ω is any 1-form on S^1, then

$$\int_{S^1} \omega = \int_0^{2\pi} h^*\omega.$$

*7. Suppose that the 1-form ω on X is the differential of a function, $\omega = df$. Prove that $\oint_\gamma \omega = 0$ for all closed curves γ on X. [HINT: Exercises 3 and 5.]

*8. Define a 1-form ω on the punctured plane $\mathbf{R}^2 - \{0\}$ by

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

(a) Calculate $\int_C \omega$ for any circle C of radius r around the origin.

(b) Prove that in the half-plane $\{x > 0\}$, ω is the differential of a function. [HINT: Try $\arctan(y/x)$ as a random possibility.]

(c) Why isn't ω the differential of a function globally on $\mathbf{R}^2 - \{0\}$?

***9.** Prove that a 1-form ω on S^1 is the differential of a function if and only if $\int_{S^1} \omega = 0$. [HINT: "Only if" follows from Exercise 6. Now let h be as in Exercise 5, and define a function g on $\mathbf{R}$ by

$$g(t) = \int_0^t h^*\omega.$$

Show that if $\int_{S^1} \omega = 0$, then $g(t + 2\pi) = g(t)$. Therefore $g = f \circ h$ for some function f on S^1. Check $df = \omega$.]

***10.** Let ν be any 1-form on S^1 with nonzero integral. Prove that if ω is any other 1-form, then there exists a constant c such that $\omega - c\nu = df$ for some function f on S^1.

11. Suppose that ω is a 1-form on the connected manifold X, with the property that $\oint_\gamma \omega = 0$ for all closed curves γ. Then if $p, q \in X$, define $\int_p^q \omega$ to be $\int_0^1 c^*\omega$ for a curve $c: [0, 1] \to X$ with $c(0) = p$, $c(1) = q$. Show that this is well defined (i.e., independent of the choice of c). [HINT: You can paste any two such curves together to form a closed curve, using a trick first to make the curves constant in neighborhoods of zero and one. For this last bit, use Exercise 4.]

***12.** Prove that any 1-form ω on X with the property $\oint_\gamma \omega = 0$ for all closed curves γ is the differential of a function, $\omega = df$. [HINT: Show that the connected case suffices. Now pick $p \in X$ and define $f(x) = \int_p^x \omega$. Check that $df = \omega$ by calculating f in a coordinate system on a neighborhood U of x. Note that you can work entirely in U by picking some $p' \in U$, for $f(x) = f(p') + \int_{p'}^x \omega$.]

13. Let S be an oriented two-manifold in $\mathbf{R}^3$, and let $\vec{n}(x) = (n_1(x), n_2(x), n_3(x))$ be the outward unit normal to S at x. (See Exercise 19 of Chapter 3, Section 2 for definition.) Define a 2-form dA on S by

$$dA = n_1\, dx_2 \wedge dx_3 + n_2\, dx_3 \wedge dx_1 + n_3\, dx_1 \wedge dx_2.$$

(Here each dx_i is restricted to S.) Show that when S is the graph of a function $F: \mathbf{R}^2 \longrightarrow \mathbf{R}$, with orientation induced from $\mathbf{R}^2$, then this dA is the same as that defined in the text.

14. Let

$$\omega = f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2$$

be an arbitrary 2-form in $\mathbf{R}^3$. Check that the restriction of ω to S is the form $(\vec{F}\cdot\vec{n})\, dA$, where

$$\vec{F}(x) = (f_1(x), f_2(x), f_3(x)).$$

[HINT: Check directly that if $u, v \in T_x(S) \subset \mathbf{R}^3$, then $\omega(x)(u, v)$ equals one half the determinant of the matrix

$$\begin{pmatrix} \vec{F}(x) \\ u \\ v \end{pmatrix}.$$

If $\vec{F}(x) \in T_x(S)$, this determinant is zero, so only the normal component of $\vec{F}(x)$ contributes.]

§5 Exterior Derivative

Forms cannot only be integrated, they can also be differentiated. We have already seen how to do this in general for 0-forms, obtaining from a smooth function f the 1-form df. In Euclidean ... to continue. If $\omega = \sum$ ...