Determinant Theorem. If $A\colon V \longrightarrow V$ is a linear map, then $A^*T = (\det A)T$ for every $T \in \Lambda^k(V)$, where $k = \dim V$. In particular, if $\phi_1, \ldots, \phi_k \in \Lambda^1(V^*)$, then

$$A^*\phi_1 \wedge \cdots \wedge A^*\phi_k = (\det A)\, \phi_1 \wedge \cdots \wedge \phi_k.$$

EXERCISES

1. Suppose that $T \in \Lambda^p(V^*)$ and $v_1, \ldots, v_p \in V$ are linearly dependent. Prove that $T(v_1, \ldots, v_p) = 0$ for all $T \in \Lambda^p(V^*)$.

2. Dually, suppose that $\phi_1, \ldots, \phi_p \in V^*$ are linearly dependent, and prove that $\phi_1 \wedge \cdots \wedge \phi_p = 0$.

3. Suppose that $\phi_1, \ldots, \phi_k \in V^*$ and $v_1, \ldots, v_k \in V$, where $k = \dim V$. Prove that

$$\phi_1 \wedge \cdots \wedge \phi_k(v_1, \ldots, v_k) = \frac{1}{k!} \det [\phi_i(v_j)],$$

where $[\phi_i(v_j)]$ is a $k \times k$ real matrix. [HINT: If the ϕ_i are dependent, then the matrix has linearly dependent rows, so Exercise 2 suffices. If not, the formula is easily checked for the dual basis in V. Now verify that the matrix does specify an alternating k-tensor on V, and use $\dim \Lambda^k(V^*) = 1$.]

4. More generally, show that whenever $\phi_1, \ldots, \phi_p \in V^*$ and $v_1, \ldots, v_p \in V$, then

$$\phi_1 \wedge \cdots \wedge \phi_p(v_1, \ldots, v_p) = \frac{1}{p!} \det [\phi_i(v_j)].$$

[HINT: If the v_i are dependent, use Exercise 1. If not, apply Exercise 3 to the restrictions $\overline{\phi_i}$ of ϕ_i to the p-dimensional subspace spanned by $v_1, \ldots, v_p$.]

5. Specifically write out Alt $(\phi_1 \otimes \phi_2 \otimes \phi_3)$ for $\phi_1, \phi_2, \phi_3 \in V^*$.

*6. (a) Let T be a nonzero element of $\Lambda^k(V^*)$, where $\dim V = k$. Prove that two ordered bases $\{v_1, \ldots, v_k\}$ and $\{v_1', \ldots, v_k'\}$ for V are equivalently oriented if and only if $T(v_1, \ldots, v_k)$ and $T(v_1', \ldots, v_k')$ have the same sign. [HINT: Determinant theorem.]

(b) Suppose that V is oriented. Show that the one-dimensional vector space $\Lambda^k(V^*)$ acquires a natural orientation, by defining the sign of a nonzero element $T \in \Lambda^k(V^*)$ to be the sign of $T(v_1, \ldots, v_k)$ for any positively oriented ordered basis $\{v_1, \ldots, v_k\}$ for V.

(c) Conversely, show that an orientation of $\Lambda^k(V^*)$ naturally defines an orientation on V by reversing the above.

7. For a $k \times k$ matrix A, let A^t denote the transpose matrix. Using the fact that det (A) is multilinear in both the rows and columns of A, prove that det (A^t) = det (A). [HINT: Use dim $\Lambda^k(\mathbf{R}^{k*}) = 1$]

8. Recall that a matrix A is *orthogonal* if $AA^t = I$. Conclude that if A is orthogonal, det $(A) = \pm 1$.

9. Let V be a k-dimensional subspace of $\mathbf{R}^N$. Recall that a basis $v_1, \ldots, v_k$ of V is *orthonormal* if

$$v_i \cdot v_j = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Let $A: V \longrightarrow V$ be a linear map, and prove the following three conditions equivalent:
(a) $Av \cdot Aw = v \cdot w$ for all $v, w \in V$.
(b) A carries orthonormal bases to orthonormal bases.
(c) The matrix of A with respect to any orthonormal basis is orthogonal.
Such an A is called an *orthogonal transformation*. [Note, by (b), it must be an isomorphism.]

*10. (a) Let V be an oriented k-dimensional vector subspace of $\mathbf{R}^N$. Prove there is an alternating k-tensor $T \in \Lambda^k(V^*)$ such that $T(v_1, \ldots, v_k) = 1/k!$ for all positively oriented ordered orthonormal bases. Furthermore, show that T is unique; it is called the *volume element* of V. [HINT: Use the determinant theorem, Exercises 8 and 9, plus dim $\Lambda^k(V^*) = 1$ for uniqueness.]
(b) In fact, suppose that $\phi_1, \ldots, \phi_k \in V^*$ is an ordered basis dual to some positively oriented ordered orthonormal basis for V. Show that the volume element for V is $\phi_1 \wedge \cdots \wedge \phi_k$. [HINT: Exercise 3.]

*11. Let T be the volume element of $\mathbf{R}^2$. Prove that for any vectors $v_1, v_2 \in \mathbf{R}^2$, $T(v_1, v_2)$ is $\pm$ one half the volume of the parallelogram spanned by v_1 and v_2. Furthermore, when v_1 and v_2 are independent, then the sign equals the sign of the ordered basis $\{v_1, v_2\}$ in the standard orientation of $\mathbf{R}^2$. Generalize to $\mathbf{R}^3$. Now how would you define the volume of a parallelepiped in $\mathbf{R}^k$?

12. (a) Let V be a subspace of $\mathbf{R}^N$. For each $v \in V$, define a linear functional $\phi_v \in V^*$ by $\phi_v(w) = v \cdot w$. Prove that the map $v \longrightarrow \phi_v$ is an isomorphism of V with V^*.

(b) Now suppose that V is oriented and $\dim V = 3$. Let T be the volume element on V. Given $u, v \in V$, define a linear functional on V by $w \longrightarrow 3!T(u, v, w)$. By part (a), there exists a vector, which we denote $u \times v$, such that $T(u, v, w) = (u \times v) \cdot w$ for all $w \in V$. Prove that this *cross product* satisfies $u \times v = -v \times u$. Furthermore, show that if $\{v_1, v_2, v_3\}$ is a positively oriented orthonormal basis for V, then $v_1 \times v_2 = v_3$, $v_2 \times v_3 = v_1$, and $v_3 \times v_1 = v_2$. (Also, $v \times v = 0$ always.)

§3 Differential Forms

In classical differential geometry, forms were symbolic quantitie that looked like

$$\sum_i f_i \, dx_i$$

$$\sum_{i<j} f_{ij} \, dx_i \wedge dx_j$$

$$\sum_{i<j<k} f_{ijk} \, dx_i \wedge dx_j \wedge dx_k.$$

These expressions were integrated and differentiated, and because experien proved anticommutativity to be convenient, they were manipulated lil alternating tensors. Modern differential forms locally reduce to the san symbolic quantities, but they possess the indispensable attribute of bei globally defined on manifolds. Global definition of integrands makes possit global integration.

Definition. Let X be a smooth manifold with or without boundary. *p-form* on X is a function ω that assigns to each point $x \in X$ an alternati p-tensor $\omega(x)$ on the tangent space of X at x; $\omega(x) \in \Lambda^p[T_x(X)^*]$.

Two p-forms ω_1 and ω_2 may be added point by point to create a r p-form $\omega_1 + \omega_2$:

$$(\omega_1 + \omega_2)(x) = \omega_1(x) + \omega_2(x).$$

Similarly, the wedge product of forms is defined point by point. If ω p-form and θ is a q-form, the $p + q$ form $\omega \wedge \theta$ is given by $(\omega \wedge \theta)(x$ $\omega(x) \wedge \theta(x)$. Anticommutativity $\omega \wedge \theta = (-1)^{pq}\theta \wedge \omega$ follows from analogous equation at each point.

0-forms are just arbitrary real-valued functions on X.

Many examples of 1-forms can be manufactured from smooth functi If $\phi: X \to \mathbf{R}$ is a smooth function, $d\phi_x : T_x(X) \to \mathbf{R}$ is a linear map at point x. Thus the assignment $x \longrightarrow d\phi_x$ defines a 1-form $d\phi$ on X, callec