P (G-P, 4.2.1). Suppose that $T \in \Lambda^p(V^*)$ and $v_1,\dots,v_p \in V$ are linearly dependent. Prove that $T(v_1,\dots,v_p) = 0$ for all T.

A: If $v_1,\dots,v_p$ are linearly dependent, then we can write one of them (after some renaming, say v_1) in terms of the others: $v_1 = \sum_{i=2}^p c_i v_i$. Then

$$\begin{aligned}
T(v_1,\dots,v_p) &= T(\sum_i c_i v_i, v_2,\dots,v_p) \\
&= \sum_{i=2}^p c_i \underbrace{T(v_i, v_2,\dots,v_i,\dots,v_p)}_{=-T(v_i,v_2,\dots,v_i,\dots,v_p)} \\
&= -\sum_{i=2}^p c_i T(v_i, v_2,\dots,v_i,\dots,v_p) \\
&= -T(v_1,\dots,v_p).
\end{aligned}$$

These are just real numbers. So $T(v_1,\dots,v_p) = -T(v_1,\dots,v_p) \implies T(v_1,\dots,v_p) = 0$. □

P (G-P, 4.2.6*). (a) Let T be a nonzero element of $\Lambda^k(V^*)$, where $\dim V = k$. Prove that two ordered bases $\{v_1,\dots,v_k\}$ and $\left\{v_1',\dots,v_k'\right\}$ for V are equivalently oriented iff $T(v_1,\dots,v_k)$ and $T(v_1',\dots,v_k')$ have the same sign. (hint: determinant theorem)

(b) Suppose that V is oriented. Show that the one-dimensional vector space $\Lambda^k(V^*)$ acquires a natural orientation, by defining the sign of a nonzero element $T \in \Lambda^k(V^*)$ to be the sign of $T(v_1,\dots,v_k)$ for any positively oriented basis $\{v_1,\dots,v_k\}$ for V.

(c) Conversely, show that an orientation of $\Lambda^K(V^*)$ naturally defines an orientation on V be reversing the above.

A: (a) Let $A : V \to V$ denote the transformation matrix between the bases $\{v_i\}$ and $\left\{v_i'\right\}$. So $A(v_i) = v_i'$ for all $i = 1,\dots,k$. By the determinant theorem,

$$T(v_1',\dots,v_k') = T(A(v_1),\dots,A(v_k)) = A^*T(v_1,\dots,v_k) = \det A \cdot T(v_1,\dots,v_k).$$

Then $\{v_i\}$ and $\left\{v_i'\right\}$ have the same orientation iff $\det A > 0$ iff $T(v_1,\dots,v_k)$ and $T(v_1',\dots,v_k')$ have the same sign, by the equation above.

(b) By (a), the sign of $T(v_1,\dots,v_k)$ is the same for any positively oriented basis $\{v_i\}$. So this defines an equivalence relation on $\Lambda^k(V^*)$. If T and S are both nonzero in $\Lambda^k(V^*)$ with the same sign, then $T(v_1,\dots,v_k)$ and $S(v_1,\dots,v_k)$ have the same sign for any positively oriented $\{v_i\}$.

(c) Define a sign on V to be $\operatorname{sgn}(\{v_1,\dots,v_k\}) := \operatorname{sgn}(T(v_1,\dots,v_k))$ for any positively oriented $T \in \Lambda^k(V^*)$.

□

P (G-P, 4.2.10*). (a) Let V be an oriented k-dimensional vector subspace of $\mathbb{R}^N$. Prove that there is an alternating k-tensor $T \in \Lambda^k(V^*)$ such that $T(v_1, \dots, v_k) = 1/k!$ for all positively oriented ordered orthonormal bases. Furthermore, show that T is unique; it is called the *volume element* of V (hint: use det theorem, ex. 8, ex. 9, plus $\dim \Lambda^k(V^*) = 1$ for uniqueness.)

(b) In fact, suppose that $\varphi_1, \dots, \varphi_k \in V^*$ is an ordered basis dual to some positively oriented ordered orthonormal basis for V. Show that the volume element for V is $\varphi_1 \wedge \dots \wedge \varphi_k$. (hint: ex. 3)

A: (a) Let $\{v_1, \dots, v_k\}$ form a positively oriented orthonormal basis for V. Let $\left\{v_1^*, \dots, v_k^*\right\} \subset V^*$ be its dual basis. Let $T := v_1^* \wedge \dots \wedge v_k^*$.

Then for any other positively oriented othonormal basis $\{u_1, \dots, u_k\}$,

$$T(u_1, \dots, u_k) = v_1^* \wedge \dots \wedge v_k^*(u_1, \dots, u_k) = \frac{1}{k!}\det[v_i^*(u_j)].$$

The matrix $[v_i^*(u_j)]$ is simply the change of basis transformation from$\{v_1, \dots, v_k\}$ to $\{u_1, \dots, u_k\}$. Since both $\{v_i\}$ and $\{u_i\}$ were orthonormal bases, by ex.9, the matrix is orthogonal. By ex.8, it has determinant ± 1. The sign just depends on the basis orientation; since both were positively oriented by choice, then the determinant is 1, and $T(u_1, \dots, u_k) = 1/k!$.

For uniqueness, realize that $v_1^* \wedge \dots \wedge v_k^* = T$ spans $\Lambda^K(V)$; ie. any other $S \in \Lambda^k(V)$ is just a multiple of T.

(b) Didn't read (b) before doing (a). Now I'm not sure if my argument is circular or not. I don't think it is. A finite dimensional space has some basis, and we showed uniqueness, so choice of basis does not matter, and choice of order doesn't matter as long as it's still positively oriented. Hopefully it checks out, but I wonder what they had in mind, because I don't think that was it.

□

P (G-P, 4.2.11*). Let T be the volume element of $\mathbb{R}^2$. Prove that for any vectors $v_1, v_2 \in \mathbb{R}^2$, $T(v_1, v_2)$ is $\pm$ one half the volume of the parallelogram spanned by v_1, v_2. Further, when v_1 and v_2 are independent, then then sign equals the sign of the ordered basis $\{v_1, v_2\}$ in the standard orientation of $\mathbb{R}^2$. Generalize to $\mathbb{R}^3$. Now how would you define the volume of a parallelepiped in $\mathbb{R}^k$?

A: Let $\varphi_1, \varphi_2 \in \Lambda(\mathbb{R}^2)$ be the dual basis vectors to $\mathbb{i} = (1,0)$ and $\mathbb{j} = (0,1)$ respectively. Then the volume element $T = \varphi_1 \wedge \varphi_2$, and if $v_1 = (a,b)$ and $v_2 = (c,d)$, we can calculate

$$\begin{aligned} T(v_1, v_2) &= \varphi_1 \wedge \varphi_2((a,b),(c,d)) \\ &= \frac{1}{2!}\det[\varphi_i(v_j)] \\ &= \frac{1}{2}\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{aligned}$$

We can remind ourselves that the area of the parallelogram spanned by two vectors (a, b) and (c, d) is precisely the determinant of the matrix they form as columns.

If v_1 and v_2 are linearly independent, then they form a new basis for the plane, and the matrix $[\varphi_i(v_j)]$ is the change of basis from $\left\{v_j\right\}$ to $\{\mathbb{i}, \mathbb{j}\}$, the standard basis (positively oriented). The sign of the determinant of the change of basis depends on the sign of the ordered basis $\{v_1, v_2\}$ with respect to the standard basis.

The volume element in $\Lambda^3(\mathbb{R}^3)$ is constructed similarly. Let $\{\varphi_1, \varphi_2, \varphi_3\}$ be the dual basis to the standard basis $\{\mathbb{i}, \mathbb{j}, \mathbb{k}\}$ of euclidean space. Then the volume element T looks like $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$. For any three vectors $v_1, v_2, v_3 \in \mathbb{R}^3$,

$$T(v_1, v_2, v_3) = \frac{1}{3!}\det[\varphi_i(v_j)].$$

This looks similar to the above, except we have a 1/3! instead of a 1/2. The determinant of the matrix is the volume of the (three-dimensional) volume of the parallelipiped formed by v_1, v_2, v_3. The factorial term, if we opt to keep it, represents taking 1/6 of the area. Strictly speaking, it is the area in euclidean space enclosed by the pyramid formed by three adjacent vertices (ie. a solid pyramid sitting in one corner of our paralleliped).

In n dimensions, we can define the volume of an n-parallelipiped simply as $\det[\varphi_i(v_j)]$, where $\{\varphi_i\}_{i=1}^n$ are the dual basis to the standard basis. $\square$