MATH 201 HOMEWORK 1

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Exercise 4.1.3:[G-P] Let $c\colon [a,b] \to X$ be a smooth curve, and let $c(a) = p$ and $c(b) = q$. Show that if ω is the differential of a function on X, $\omega = df$, then

$$\int_a^b c^*\omega = f(q) - f(p).$$

Proof. By hypothesis, we have $\omega = df$ for some function f on X. Then since pullbacks commute with the derivative, one has

$$c^*\omega = c^*(df) = d(c^*f) = d(f \circ c).$$

Then by the fundamental theorem of calculus, one must have

$$\int_a^b c^*\omega = \int_a^b d(f \circ c) = f(c(b)) - f(c(a)) = f(q) - f(p),$$

as desired. □

Exercise 4.2.1:[G-P] Suppose that $T \in \Lambda^p(V^*)$ and $v_1, \dots, v_p$ are linearly dependent. Prove that $T(v_1, \dots, v_p) = 0$ for all $T \in \Lambda^p(V^*)$.

Proof. Pick any $T \in \Lambda^p(V^*)$. If $v_1, \dots, v_p$ are linearly dependent, then there exist constants $a_1, \dots, a_p$, not all of which are zero, such that $a_1v_1 + \cdots + a_pv_p = 0$. Without loss of generality, we may assume that $a_1 \neq 0$. Then one has $v_1 = b_2v_2 + \cdots + b_pv_p$, where $b_i = a_i/a_1$ for each $2 \leq i \leq p$. In particular, since T is alternating, one must have

$$\begin{aligned} T(v_1, \dots, v_p) &= T(b_2v_2 + \cdots + b_pv_p, v_2, \dots, v_p) \\ &= b_2T(v_2, v_2, \dots, v_p) + \cdots + b_pT(v_p, v_2, \dots, v_p) \\ &= 0 + \cdots + 0 \\ &= 0. \end{aligned}$$

□

Exercise 4.2.2: Dually, suppose that $\phi_1, \dots, \phi_p \in V^*$ are linearly dependent, and prove that $\phi_1 \wedge \cdots \wedge \phi_p = 0$.

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Proof. If $\phi_1, \ldots, \phi_p$ are linearly dependent, then there exist constants $a_1, \ldots, a_p$, not all of which are zero, such that $a_1\phi_1 + \cdots + a_p\phi_p = 0$. Without loss of generality, suppose that $a_1 \neq 0$. Then $\phi_1 = b_2\phi_2 + \cdots + b_p\phi_p$ where $b_i = a_i/a_1$ for all $2 \leq i \leq p$. Then by the properties of the wedge product, one must have

$$\begin{aligned}
\phi_1 \wedge \cdots \phi_p &= (b_2\phi_2 + \cdots b_p\phi_p) \wedge \phi_2 \cdots \phi_p \\
&= b_2(\phi_2 \wedge \phi_2 \wedge \cdots \wedge \phi_p) + b_3(\phi_3 \wedge \phi_2 \wedge \cdots \wedge \phi_p) + \cdots + b_p(\phi_p \wedge \phi_2 \wedge \cdots \wedge \phi_p) \\
&= b_2(0) + b_3(0) + \cdots + b_p(0) \\
&= 0.
\end{aligned}$$

□