Manifolds II HW 1

Brian Ma

[Sj] Exercise 2.1 Consider the forms $\alpha = x\ dx - y\ dy$, $\beta = z\ dx \wedge dy + x\ dy \wedge dz$, $\gamma = z\ dy$ on $\mathbb{R}^3$. Calculate:

(i) $\alpha\beta$, $\alpha\beta\gamma$;

Proof: We have

$$\begin{aligned}\alpha\beta &= (x\ dx - y\ dy)(z\ dx \wedge dy + x\ dy \wedge dz)\\ &= xy\ dx \wedge dx \wedge dy - yz\ dy \wedge dx \wedge dy + xz\ dx \wedge dx \wedge dy - x^2\ dx \wedge dy \wedge dz\\ &= -x^2\ dx \wedge dy \wedge dz\end{aligned}$$

Also, we have

$$\alpha\beta\gamma = (-x^2\ dx \wedge dy \wedge dz)(z\ dy) = 0$$

(ii) $d\alpha, d\beta, d\gamma$.

Proof: We have

$$\begin{aligned}d\alpha &= (\frac{\partial x}{\partial x}dx + \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz) \wedge dx - (\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial y}dy + \frac{\partial y}{\partial z}dz) \wedge dy\\ &= dx \wedge dx - dy \wedge dy\\ &= 0\end{aligned}$$

Similarly,

$$d\beta = dz \wedge dx \wedge dy + dx \wedge dy \wedge dz = 2\ dx \wedge dy \wedge dz$$

and

$$d\gamma = dz \wedge dy$$

□

[Sj] Exercise 2.2 Compute the exterior derivative of the following forms. Recall that a hat indicates that a term has to be omitted.

(i) $e^{xy+z^2}\ dx$

Proof: We have

$$\begin{aligned} d(e^{xy+z^2}\ dx) &= (\frac{\partial(e^{xy+z^2})}{\partial x}dx + \frac{\partial(e^{xy+z^2})}{\partial y}dy + \frac{\partial(e^{xy+z^2})}{\partial dz}dz) \wedge dx \\ &= xe^{xy+z^2}\ dy \wedge dx + 2ze^{xy+z^2}\ dz \wedge dx \end{aligned}$$

(ii) $\sum_{i=1}^n x_i^2\ dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

Proof: Let $\alpha = \sum_{i=1}^n x_i^2\ dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. We have

$$\begin{aligned} d\alpha &= \sum_{i=1}^n (\frac{\partial(x_i^2)}{\partial x_1}dx_1 + \cdots + \frac{\partial(x_i^2)}{\partial x_n}dx_n) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^n (2x_i\ dx_i) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \end{aligned}$$

Note that when i is odd, we interchange the variables an even number of times and when i is even, we interchange the variables an odd number of times.

Thus, by the alternating property, the terms at an odd index will be positive while the terms at an even index will be negative. So,

$$d\alpha = (2x_1 - 2x_2 + 2x_3 - \cdots 2x_n)\ dx_1 \wedge \cdots \wedge dx_n$$

□

[Sj] Exercise 2.5: Write the coordinates on $\mathbb{R}^{2n}$ as $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$. Let

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n = \sum_{i=1}^n dx_i \wedge dy_i$$

Compute $\omega^n = \omega\omega\cdots\omega$ (n-fold product). (First work out the cases for $n = 1, 2, 3$)

Proof: For $n = 1$, we have $\mathbb{R}^2$ with coordinates (x_1, y_1). Thus,

$$\omega^1 = \omega = dx_1 \wedge dy_1$$

For $n = 2$, we have $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2). So, $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Then,

$$\begin{aligned} \omega &= dx_2 \wedge dy_2 \wedge dx_1 \wedge dy_1 + dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \\ &= 2\ dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \end{aligned}$$

For $n = 3$, we have $\mathbb{R}^6$ with coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$. Let $z_1 = dx_1 \wedge dy_1, z_2 = dx_2 \wedge dy_2, z_3 = dx_3 \wedge dy_3$. Then, $\omega = z_1 + z_2 + z_3$.

Notice that $a \wedge b = b \wedge a$ for all $a, b \in \{z_1, z_2, z_3\}$. Then,

$$\begin{aligned}
\omega &= (z_1 + z_2 + z_3)(z_1 + z_2 + z_3)(z_1 + z_2 + z_3) \\
&= (2\ z_1 \wedge z_2 + 2\ z_1 \wedge z_3 + 2\ z_2 \wedge z_3)(z_1 + z_2 + z_3) \\
&= 2\ z_1 \wedge z_2 \wedge z_3 + 2\ z_1 \wedge z_2 \wedge z_3 + 2\ z_1 \wedge z_2 \wedge z_3 \\
&= 6\ z_1 \wedge z_2 \wedge z_3 \\
&= 6\ dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3
\end{aligned}$$

Thus, for $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, we get

$$\omega^n = n!\ dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$$

$\square$