1. Let $T \in \Lambda^p(V^*)$, $v_1 \cdots v_p \in V$ are linearly dependent.
Thus, $\exists j$ s.t $v_j = \sum_{\substack{i=1 \\ i \neq j}}^{p} a_i v_i$

$\Rightarrow T(v_1, \cdots v_p) = T(v_1, \cdots, a_1 v_1 + \cdots + a_p v_p, \cdots v_p)$
$= a_1 T(v_1 \cdots, v_1, \cdots v_p) + \cdots + a_p T(v_1, \cdots, v_p, \cdots v_p)$ by multilinearity.
$= 0$.

2. Suppose $\phi_1, \dots \phi_p \in V^*$ are linearly dependent, then $\exists j$ s.t. $\phi_j = a_1\phi_1 + \dots + a_p\phi_p$, jth term omitted.

Since $\phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$, $\phi_i \wedge \phi_i = 0$

then $\phi_1 \wedge \cdots \wedge (a_1\phi_1 + \dots + a_p\phi_p) \wedge \cdots \wedge \phi_p$

$= a_1\phi_1 \wedge \cdots \wedge \phi_1 \cdots \wedge \phi_p + \dots + \phi_1 \wedge \cdots \wedge a_p\phi_p \wedge \cdots \wedge \phi_p$

$= 0.$

3. Suppose $\phi_1, \cdots \phi_k \in V^*$, $v_1, \cdots v_k \in V$, $k = \dim V$.

$$\phi_1 \wedge \cdots \wedge \phi_k = Alt(\phi_1 \otimes \cdots \otimes \phi_k)$$

$$= \frac{1}{k!} \sum_{\delta \in S_k} \phi_{\delta(1)} \otimes \cdots \otimes \phi_{\delta(k)} \cdot (-1)^{sgn(\delta)}.$$

$$\phi_1 \wedge \cdots \wedge \phi_k(v_1 \cdots v_k) = \frac{1}{k!} \sum_{\delta \in S_k} (-1)^{sgn(\delta)} \phi_{\delta(1)} \otimes \cdots \otimes \phi_{\delta(k)} (v_1 \cdots v_k)$$

$$= \frac{1}{k!} \sum_{\delta \in S_k} (-1)^{sgn(\delta)} \phi_{\delta(1)}(v_1) \cdots \phi_{\delta(k)}(v_k).$$

Define a matrix $M = [\phi_i(v_j)]$

This is the expression of $det(M)$

$\Rightarrow \phi_1 \wedge \cdots \wedge \phi_k(v_1 \cdots v_k) = \frac{1}{k!} det[\phi_i(v_j)]$.

4. The proof for 3 is true for all $1 \leq k \leq dim(V)$, thus, it's proven.

6. let $T \in \Lambda^k(V^*)$, $dim V = k$, let $\underset{B_1}{\{v_1 \cdots v_k\}}$, $\underset{B_2}{\{v_1', v_2' \cdots v_k'\}}$ be two ordered basis.
Then, $\exists A \in Hom(V, V)$ s.t A transform B_1 to B_2.
$\Rightarrow$ Since B_1, B_2 are equivalently oriented, then $|A| > 0$.

Consider $T(v_1', \cdots v_n') = T(Av_1, \cdots Av_n)$
$= A^* T(v_1 \cdots v_n)$
$= (det A) T(v_1, \cdots v_n)$ by determinant theorem.
$\Rightarrow T(v_1 \cdots v_n), T(v_1' \cdots v_n')$ have same sign

$\Leftarrow$ Assume $T(v_1 \cdots v_k), T(v_1' \cdots v_k')$ have the same sign
Then $T(v_1', \cdots v_k') = \lambda T(v_1 \cdots v_k)$ $\lambda > 0$.
let A be the transformation matrix from $B_1 \rightarrow B_2$.
$T(v_1', \cdots v_n') = (det A) T(v_1 \cdots v_n) = \lambda T(v_1 \cdots v_n)$
$\Rightarrow det A = \lambda > 0$
$\Rightarrow B_1, B_2$ are same oriented.

b) Let V be an oriented vector space. Define a natural orientation of $T \in \Lambda^k(V^*)$ to be the sign of $T(v_1 \cdots v_k)$ for any positively oriented ordered basis $\{v_1 \cdots v_k\}$ for V.

By (a), we prove the sign of $T(v_1 \cdots v_k)$ is independent the choice of $\{v_1 \cdots v_k\}$ when two basis has same orientation.

Thus, define the relation $\sim$: $T(V_1, \cdots V_n)$'s sign. $\{V_1 \cdots V_n\}$ is positively oriented.
Reflexity obviously hold.
symmetricity obviously hod.
Transitivity obviously hold
$\Rightarrow \sim$ is an equivalence relation
$\Rightarrow$ It's a well-defined orientation
(b) ~~Define the orientation $\Lambda^k(V^*)$ as~~ (b).
In fact,
(c) Let $\Lambda^k(V^*)$ be oriented, let $T \in \Lambda^k(V^*)$ $T \neq 0$.
Define an orientation to V as:
let $\{V_1, \cdots V_k\} = B_1$, $\{V_1', \cdots V_k'\} = B_2$.
if $T(V_1, \cdots V_k)$! $T(V_1', \cdots V_k')$ has same sign, then by (a), $\{V_1 \cdots V_k\}$, $\{V_1' \cdots V_k'\}$ are equivalently oriented.
Fix $\{e_1, \cdots e_k\}$ and all basis equivalently oriented to $\{e_1, \cdots e_k\}$ can be assigned $+1$.
Otherwise -1.
$\Rightarrow$ It defines a natural orientation on V.
10*. (a) let V be an oriented k-dimensional vector subspace of $\mathbb{R}^n$.
let $T \in \Lambda^k(V^*)$.
Let $\{V_1, \cdots V_k\}$, $\{V_1', \cdots V_k'\}$ be two sets of positively oriented ordered orthonormal bases.
Then $\exists$ A be the transitional map.
by 9. A is orthogonal, by 8. $\det(A) = 1$.
by determinant Theorem. $A^*T = T$.
So $T(V_1 \cdots V_k) = A^*T(V_1 \cdots V_k) = T(AV_1, \cdots AV_k) = T(V_1', \cdots V_k')$.
This proves that, $T \in \Lambda^k(V^*)$ will map positively oriented orthogonal vectors to the same value.
It suffices to show, $\exists T \in \Lambda^k(V^*)$ s.t $T(V_1 \cdots V_k) = \frac{1}{k!}$
Let $\phi_1 \cdots \phi_k$ be the dual vector of $V_1 \cdots V_k$.
Consider $\phi_1 \wedge \cdots \wedge \phi_k = \frac{1}{k!} \sum_{\delta \in S_k} (-1)^{\text{sgn}(\delta)} \phi_{\delta(1)} \otimes \cdots \otimes \phi_{\delta(k)} (V_1 \cdots V_k)$

By pairing, only $id \in S_k$ will give the nonzero value

$= \frac{1}{k!}$

(b) Since $\dim \Lambda^k(V^*) = 1$, then ~~[illegible]~~ T is unique. Since any $T' = \lambda T$

b) 1. already prove in part (a).

F2.1. let L be a n-dim space. let α be a p-vector, $\alpha \neq 0$. Let M_α be the subspace of L consisting of all vectors δ satisfying $\alpha \wedge \delta = 0$.

$\alpha = \sum a (v_1 \wedge \cdots \wedge v_p)$

assume $\dim(M_\alpha) > p$, then $\exists\ v_{p+1} \in M_\alpha$ s.t v_{p+1} is linearly independent from $v_1 \cdots v_p$.

Then $a(v_1 \wedge \cdots \wedge v_p) \wedge v_{p+1} \neq 0$.

$\Rightarrow \alpha \neq 0$.

Contradiction

$\Rightarrow \dim(M_\alpha) \leq p$.

$\Rightarrow$ if $\dim(M_\alpha) = p$. let $v_1 \cdots v_p$ be the basis of M_α. Extend $v_1 \cdots v_p$ to a basis of L $v_1 \cdots v_n$.

Then $\alpha = \sum a(v_1 \wedge \cdots v_p)$

$\cancel{\alpha \wedge v_i = \sum a(v_1 \wedge \cdots \wedge v_p) \wedge v_i} = 0$.

if $\alpha \neq v_1 \wedge \cdots \wedge v_p$, then $\exists$ a component $\underbrace{v_1 \wedge \cdots \wedge v_p \wedge \cdots \wedge v_m}_{p}$ $n \geq m > p$

Thus $\exists\ v_i \in \{v_1 \cdots v_p\}$ s.t

$v_1 \wedge \cdots \wedge v_p \wedge \cdots \wedge v_m \wedge v_i \neq 0$.

Contradiction

$\Rightarrow \alpha = v_1 \wedge \cdots \wedge v_p$.

$\Leftarrow$ let $\alpha = v_1 \wedge \cdots \wedge v_p$, then $\dim(M_\alpha) \leq p$ by the first part of proof.

but notice that $\alpha \wedge v_i = 0 \quad \forall 1 \leq i \leq p$

$\Rightarrow \dim(M_\alpha) \geq p$

$\Rightarrow \dim(M_\alpha) = p$.

2. let α be any $(n-1)$vector, then $\dim(M_\alpha) \leq n-1$ $\qquad$ α_i, α_j omit

For 2 cases, $\alpha = \alpha_1 \wedge \cdots \wedge \underset{\text{omit}}{\hat{\alpha}_i} \wedge \cdots \wedge \alpha_n + \alpha_1 \wedge \cdots \wedge \hat{\alpha}_j \wedge \cdots \wedge \alpha_n = (\pm)\alpha_1 \wedge \cdots \wedge \alpha_n \wedge (\alpha_i + \alpha_j)$

$= \alpha_1 \wedge \cdots \wedge \alpha_n \wedge \alpha_k$ $\qquad$ redenote as α_k