

## MATH 209, MANIFOLDS II, WINTER 2011

### Homework Assignment V: Orientations and integration

1. Let  $F: S^n \rightarrow S^n$  be the antipodal map.
  - (a) Prove that  $F$  is orientation preserving when  $n$  is odd and orientation reversing when  $n$  is even.
  - (b) Prove that  $\mathbb{R}P^n$  is orientable if and only if  $n$  is odd.
2. Let  $F: \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function. Show that  $F$  is necessarily orientation preserving at its regular points, i.e.,  $F^*dx \wedge dy = f dx \wedge dy$  with  $f \geq 0$ .
3. Let  $N$  be a hypersurface in  $M$  and let  $\omega$  be a volume form on  $M$ .
  - (a) Let  $v$  be a vector field nowhere tangent to  $N$ . Prove that  $i_v\omega|_N$  is a volume form on  $N$ .
  - (b) Prove that  $i_v\omega|_N = i_w\omega|_N$  if  $v - w$  is tangent to  $N$ .

Remark. Assume that the hypersurface  $N$  is the boundary of  $M$ . Then the construction of Part (a) gives an alternative description of the orientation induced on  $N$ . Indeed, let  $v$  point outward and let an orientation of  $M$  be determined by  $\omega$ . Then the induced orientation of  $N = \partial M$  is determined by  $i_v\omega$  and is well defined. Note also that in both (a) and (b) it suffices to have  $v$  defined only along  $N$ .

4. Let  $M \subset \mathbb{R}^3$  be the graph of a function  $z = f(x, y)$  with  $(x, y)$  lying in some bounded closed domain  $U$  of  $\mathbb{R}^2$ . Let  $\mathbf{v}$  be a vector field in  $\mathbb{R}^3$  defined on a neighborhood of  $M$ . Recall from vector calculus that the surface integral of  $\mathbf{v}$  over  $M$  is defined as

$$\iint_M \mathbf{v} \cdot d\mathbf{S} = \iint_M \mathbf{v} \cdot \mathbf{n} dS = \iint_U (\mathbf{v} \cdot \mathbf{n})(x, y) \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dx dy,$$

where  $\mathbf{n}$  is the unit upward normal vector field to  $M$ . Let  $\omega = dx \wedge dy \wedge dz$ .

- (a) Prove that  $\iint_M \mathbf{v} \cdot d\mathbf{S} = \int_M i_{\mathbf{v}}\omega$ , where the orientation of  $M$  is induced by  $\omega$  and  $\mathbf{n}$  as in Problem 3.
- (b) Prove that  $F^*i_{\mathbf{n}}\omega = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dx \wedge dy$ , where  $F: U \rightarrow M$  is the natural diffeomorphism  $(x, y) \mapsto (x, y, f(x, y))$ .

Remark. Since every hypersurface is locally a graph (in some orthogonal coordinates), this statement indicates that in general the integral of  $\mathbf{v}$  over a hypersurface in the sense of vector calculus is equal to the integral of  $i_{\mathbf{v}}\omega$ .

5. Problem 15-3, page 407 of the textbook.
- 6\*. Prove the fundamental theorem of algebra as outlined in class.