[HW ON: de Rham coho. Degree. Pull-backs. Pushforwards.].

1 Prove that the de Rham cohomology of the real line is given by $H^0(\mathbb{R}) = \mathbb{R})$, $H^1(\mathbb{R}) = 0$.

2 Prove by hand that the de Rham cohomology of the circle is $H^0(S^1) = S^1$, $H^1(S^1) = \mathbb{R}$.

3. Let $f(z)$ be a holomorphic function of the complex variable $z = x + iy$. Then we can think of f as a smooth map $\mathbb{R}^2 \to \mathbb{R}^2$ by writing $w = u + iv = f(z)$ and observing that $u = u(x, y), v = v(x, y)$ are smooth functions of (x, y). Let $dz = dx + idy$ be the complex valued 1-form.

Prove that $f^* dz = f'(z) dz$ where $f'(z)$ is the complex derivative of f.

4. Let $S^1 \subset \mathbb{C}$ be the standard unit circle, oriented the standard way, and endowed with its standard 'volume form' $\mu =$ '$d\theta$', so that $\int_{S^1} \mu = 2\pi$. For n an integer let $f_n : S^1 \to S^1$ be the restriction of $z \to z^n$ to the unit circle. **Show that** $f^* \mu = n\mu$**.**

5. Let $F : \mathbb{C} = \mathbb{R}^2 \to S^2$ be the stereographic projection chart, the inverse of the stereo projection map $S^2 \setminus N \to \mathbb{R}^2$ where $G(x, y, z) = (x/(1-z), y/(1-z)) = u + iv \in \mathbb{C}$. Compute, in the u, v coordinates, the pull-back by F of the standard 'volume form' on the sphere.

6. Identifying $\mathbb{C} \cup \{\infty\}$ with S^2 as above, we see that any complex polynomial $p(z) = a_d z^d + \ldots + a_1 z + a_0$ can be viewed as a map $S^2 \to S^2$ which happens to map $N = \infty$ to itself.

a) Prove that the degree of the map z^d is d, for d a positive integer.

b) Prove that the degree of any polynomial of degree d is d by finding a homotopy from p to the map of part (a).

7. The $n+1$ sphere is (topologically) the suspension of the n-sphere. Use this fact and induction to prove that for any d there is a map $S^n \to S^n$ having degree d.

8. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the standard projection $\pi(x, y, z) = (x, y)$.

a) Show that if β is one-form on $\mathbb{R}^3$ which can be written as $\beta = \pi^* \alpha$ where α is a one-form on $\mathbb{R}^2$, then β has the form $\beta = f(x, y)dx + g(x, y)dy$.

b) Show tha if X is a vector field on $\mathbb{R}^3$ and if $\pi_* X$ is defined as a a vector field on $\mathbb{R}^2$, then X must be of the form $X = f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial z} + h(x, y, z)\frac{\partial}{\partial z}$.

[Hint: It may help to look up the definition of π-projectible vector field. Warner is one reference containing this notion.]

9. The Poincaré metric ds^2 on the upper half plane is $\frac{dx^2 + dy^2}{y^2}$

a) Referring back to exercise **1**, show that $ds^2 = \frac{|dz|^2}{Im(z)}$

b) Let $f(z) = (az + b)/(cz + d)$. Show that

$$f^* ds^2 = \frac{1}{ad - bc} ds^2$$

provided a, d, b, c are real.