MATH 209, Last HW, WINTER 2020

Mostly, basic symplectic geometry.

Below S denotes a symplectic manifold with symplectic form ω. Darboux coordinates on S are coordinates q^i, p_i such that $\omega = \Sigma dq^i \wedge dp_i$. The canonical one form θ on $S = T^*Q$ is given in coordinates by $\Sigma p_i dq^i$, has a nice intrinsic definition, and satisfies $\omega = -d\theta$.

a) Define the 'cotangent lift' of a diffeo $Q \to Q$ by $T^*\phi(q,p) = (\phi(q), (d\phi_q)^{-1*}p)$ for $p \in T_q^*Q$. Show that the cotangent lift of any diffeo preserves the canonical one-form.

b) Find a symplectic transformation of $T^*\mathbb{R}$ which is not the cotangent lift of a diffeo of $\mathbb{R}$.

c). The Hamiltonian vector field associated to a function H on a symplectic manifold S, also called its symplectic gradient, is defined by

$$dH = \omega(X_H, \cdot)$$

.

Show that in Darboux coordinates, the ODE $\dot{\gamma} = X_H(\gamma)$ reads

$$\ddot{q}^i = \frac{\partial H}{\partial p_i}$$

$$\ddot{p}_i = -\frac{\partial H}{\partial q^i}$$

This system of ODEs is called "Hamilton's equations" for the Hamiltonian H.

d) Verify that the flow of a Hamiltonian vector field preserves the symplectic form:

$$L_{X_H}\omega = 0$$

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POISSON BRACKETS

e) The Poisson bracket of two functions on a symplectic manifold is defined by $\{f, g\} = \omega(X_f, X_g)$. Verify that the map

$$f, g \mapsto \{f, g\}$$

gives the vector space of smooth functions on S the structure of a Lie algebra. Verify that also

$$\{f, gh\} = g\{f, h\} + h\{f, g\}$$

for any three smooth functions f, g, h.

g) Compute the Poisson bracket $\{f, g\}$ in Darboux coordinates.

h) With X_H as above, verify that $X_H[f] := df(X_H) = \{f, H\}$. Conclude that Hamilton's equations is equivalent to the evolution equation $\frac{df}{dt} = \{f, H\}$ valid for all smooth functions f.

i) Verify that the map $H \mapsto X_H$ is either a Lie-algebra homomorphism or anti-homomorphism, between the Lie algebra of functions on S under Poisson bracket, and the Lie algebra of vector fields which preserve the symplectic form, under Lie bracket of vector fields.

CO-TANGENT SPECIALS

j) For X a vector field on Q write $P_X : T^*Q \to \mathbb{R}$ for the same vector field, but viewed via duality as a fiber-linear function on T^*Q. Thus:

$$P_X(q,p) = p(X(q)).$$

Verify that the Hamiltonian flow of the Hamiltonian vector field for P_X is the cotangent lift of the flow of X on Q.

k) Verify that $\{P_X, P_Y\} = -P_{[X,Y]}$ for vector fields X, Y on Q.

Verify that

$$\{P_X, \pi^* f\} = \pi^* X[f]$$

where $\pi : T^*Q \to Q$ is the projection and f an arbitrary smooth function on Q.

Verify that $\{\pi^* f, \pi^* g\} = 0$

FUBINI-STUDY;

L) As we have seen in lectures , $\mathbb{C}P^n = \mathbb{P}(\mathcal{H})$ is a symplectic manifold whose symplectic form, called the Kahler form, is induced by the symplectic form on $\mathcal{H}$ which is the imaginary part of the Hermitian inner product. Here $\mathcal{H}$ is a Hilbert space of dimension $n+1$.

L1) For $A : \mathcal{H} \to \mathcal{H}$ a self-adjoint operator, verify that $H([\psi]) = \langle \psi, A\psi \rangle / \langle \psi, \psi \rangle$ is a smooth real valued function on $\mathbb{C}P^n$.

L2) Use the spectral theorem to relate the eigenspaces of A to the fixed points of the Hamiltonian vector field X_H.

L3) For the case $dim(\mathcal{H}) = 2$ and A having two distinct eigenvalues show that the flow of X_H corresponds to rotation of the sphere $S^2 = \mathbb{C}P^1$. Find a formula relating the period of this rotation to the difference of the two eigenvalues.