MATH 209, MANIFOLDS II, WINTER 2011

Homework Assignment I: One-forms and integration

1. [HAND IN] A coframe for a manifold is a collection of one-forms which everywhere frames T^*M. Suppose that $\{\theta^i\}_{i=1,\dots,n}$ coframes M. Expand out $d\theta^i$ in terms of the coframe to define functions c^i_{jk} be $d\theta^i = \Sigma c^i_{jk}\theta^j \wedge \theta^k$.

a) Show that if we impose $c^i_{jk} = -c^i_{kj}$ then these functions are uniquely determined by the coframe.

b) Use $d^2 = 0$ to show that if these functions are all constants then they are the structure constants of some Lie algebra.

2 Prove the validity of

Theorem 0.1 (Cartan's magic formula)**.**

$$L_X = di_X + i_X d$$

as operators on the algebra Ω of smooth forms on a manifold , where L_X is the operation of Lie differentiation by a smooth vector field X on that manifold.

Use $L_X = \frac{d}{dt}|_{t=0}\Phi_t^*$ where ϕ_t is the flow of X as your definition of the operator $L_X : \Omega \to \Omega$. For the purposes of this exercise, set $S_X = di_X + i_X d$.

Step 0. Verify that $L_X f = S_X f$ for f a zero form, which is to say, a smooth function.

Step 1. Verify that $L_X d = dL_X$ and similarly that $S_X d = dS_X$.

Step 2. Verify that both L_X and S_X are algebra homomorphisms: that is, for any $\alpha, \beta \in \Omega$ we have that $L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$ and likewise for S_X.

Step 3. Finish up.

3. Prove the identities

$$\begin{aligned} L_X L_Y - L_Y L_X &= L_{[X,Y]} &(1)\\ L_X i_Y - i_Y L_X &= i_{[X,Y]} &(2) \end{aligned}$$

where X, Y are smooth vector fields on a manifold and L_X, L_Y, i_X, i_Y the corresponding operators on the algebra Ω of forms.