MATH 209, MANIFOLDS II, WINTER 2014

Homework Assignment 3: Differential Forms

1. [HAND IN] The following exercise relates div, $grad$ and $curl$ of vector calculus to the exterior derivative on $M = \mathbb{R}^3$. Equip $\mathbb{R}^3$ with the standard inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{X}$ be the space of smooth vector fields on $\mathbb{R}^3$ and Ω^0 be the space of smooth functions thereon. Define isomorphisms

- $\Psi_1 \colon \mathcal{X} \to \Omega^1(\mathbb{R}^3)$ by $\Psi_1(v) = \langle v, \cdot \rangle$.
- $\Psi_2 \colon \mathcal{X} \to \Omega^2(\mathbb{R}^3)$ by $\Psi_2(v) = i_v(dx \wedge dy \wedge dz)$.
- $\Psi_3 \colon \Omega^0 \to \Omega^3(\mathbb{R}^3)$ by $\Psi_3(f) = f\, dx \wedge dy \wedge dz$.

Prove that the following diagram commutes (up to signs):

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{grad} & \mathcal{X} & \xrightarrow{curl} & \mathcal{X} & \xrightarrow{div} & C^\infty(\mathbb{R}^3) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} & & \downarrow{\scriptstyle \Psi_3} \\
C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

2. Rewrite Maxwell's equations in terms of exterior derivatives. The electromagnetic field is a two form $F = \Sigma F_{\mu\nu} dx^\mu \wedge dx^\nu$, where the indices μ, ν run through $0, 1, 2, 34$ and the coordinates of space-time are x^0, x^1, x^2, x^3 with $x^0 = ct$ being the time coordinate. Write i, j for any of the spatial indices $1, 2, 3$. The components F_{0i} correspond to the electric field and F_{ij} to the magnetic field. The charge- current density is a one-form $J = \Sigma J_\mu dx^\mu$ with J_0 corresponding to charge density ρ.

You will have to learn about the Hodge star operator for Minkowski space.

3. Prove that for a vector field v on M and $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$, we have $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v\beta)$.

4. A smooth map $F \colon M \to N$ induces the algebra homomorphism $F^* \colon \Omega^*(N) \to \Omega^*(M)$. Prove that $dF^* = F^*d$.