MATH 209, MANIFOLDS II, WINTER 2011

Homework Assignment III: Linear Algebra

1. [HAND IN] Let $\alpha_1, \dots, \alpha_n$ be elements of V^*. Define the matrix a_{ij} by $\alpha_i = \sum_j a_{ij} e_j^*$. Prove that $\alpha_1 \wedge \dots \wedge \alpha_n = \det(a_{ij})\, e_1^* \wedge \dots \wedge e_n^*$.

2. Let $F\colon V \to V$ be a linear map. Prove that the map of one-dimensional vector spaces $F^*\colon \bigwedge^n V^* \to \bigwedge^n V^*$ is multiplication by $\det F$.

3. An element $\omega \in A^2(V) = \bigwedge^2 V^*$ is called *non-degenerate* or a *linear symplectic form* on V if $i_v\omega \neq 0$ for any non-zero $v \in V$. Define the matrix $A = (a_{ij})$ by $\omega = \sum_{i,j} a_{ij}\, e_i^* \otimes e_j^*$. Note that A is skew-symmetric, i.e., $A^\top = -A$, since ω is skew-symmetric.

(a) Consider the map $V \to V^*$ given by $v \mapsto i_v\omega$. Prove that this map is an isomorphism if and only if ω is non-degenerate. Furthermore, A the matrix of this map in the bases $\{e_i\}$ and $\{e_i^*\}$. Thus, ω is non-degenerate if and only if $\det A \neq 0$.

(b) If ω is symplectic prove that $dim V$ is even.

(c) With $n = 2m = dim(V)$ as in (b), prove that ω is non-degenerate if and only if $\omega^m \neq 0$ in $\bigwedge^n V^*$.

(d) Let $\mu = e_1^* \wedge e_2^* \wedge \ldots \wedge e_n^*$ be the standard volume form, and write $\omega^m = P\mu$ with $P \in \mathbb{R}$ and $2m = dim(V)$ as in item (c). Show that $P^2 = det(A)$.

Remark. The function $P = P(A)$ of (d) is called the "Pfaffian". It is a polynomial: an analytic square root for the determinant acting on skew-symmetric matrices. It appears in the Chern-Gauss-Bonnet formula.

4. Consider the map $\Phi\colon V \otimes W^* \to L(W, V)$ sending $v \otimes \alpha$, where $v \in V$ and $\alpha \in W^*$, to the linear map $W \ni x \mapsto \alpha(x)v \in V$. Prove that Φ, define a linear isomorphism.