MATH 209, MANIFOLDS II, WINTER 2011

Homework Assignment 2: More one-forms and integration

[Contact forms; integral surfaces; two-forms].

PRELUDE: You will need to have done exercise 4 from from HW 1, the exercise about $\int xdy$, in order to do the exercise to hand in.

1. [HAND IN] An integral curve c for a one-form α is a curve satisfying $c^*\alpha = 0$. Let the curve $c(t) = (x(t), y(t), z(t)), 0 \leq t \leq 1$be an integral curve for $\alpha = dz - ydx$. Suppose that the projection $C(t) = (x(t), y(t))$ of c to the xy plane is a closed curve. Show that $z(1)?z(0)$ is the area enclosed by C. [Cf. problem 2 of HW 1]

2. [HAND IN] A non-vanishing one-form α on a manifold M defines a hyperplane field $D \subset TM$ by $D = \{\alpha = 0\}$, in other words $D_m = \{v \in T_mM : \alpha(m)v = 0\} \subset T_mM$ for $m \in M$. An integral submanifold for this field is an embedded submanifold whose tangent spaces at each point is contained in the hyperplane field at that point.

(a) Let $i : \Sigma \to M$ be the inclusion of a submanifold into M. Show that Σ is an integral submanifold if and only if $i^*\alpha = 0$.

(b) **Show that the hyperplane field defined by** $\alpha = dz - ydx$ **on** $M = \mathbb{R}^3$ **admits no integral surfaces.**

Hint: Argue by contradiction. Suppose it did and let Σ be such a surface and i the inclusion. Argue that Σ can be expressed as a graph $z = f(x, y)$ and then look at closed curves lying on Σ and use problem **1** .

3. Let α be a non-vanishing one-form

a) Show that if α admits an integral hypersurface then $d\alpha \equiv 0 \pmod{\alpha}$.
Hint: let i be the inclusion. Use $i^*d = di^*$.

b) Show that a two-form ω is 0 mod α if and only if $\omega \wedge \alpha = 0$.

4. Write $z = x + iy$ so as to identify $\mathbb{C}$ with $\mathbb{R}^2$ as usual. Express the real and imaginary parts of the complex one-form dz/z in terms of x, y, dx, dy. Repeat, using polar coordinates $r, ?$ and $dr, d?$.

5. For $\omega = dx \wedge dy + dz \wedge dw$ on $\mathbb{R}^4$ compute $\omega^2 := \omega \wedge \omega$.

6. Coordinatize $\mathbb{R}^{2n}$ by coordinates $x^1, \ldots, x^n, y_1, \ldots y_n$. Set $\omega = \Sigma dx^i \wedge dy_i$. Then $\omega^n = cdx^1 \wedge dx^2 \ldots \wedge dx^n \wedge dy^1 \wedge \ldots dy^n$ for some constant c. Find c.