More on connections , specific to projective space and the Fubini-Study metric and the canonical line bundle.

Background and basics: section 1.2 of my paper "Heisenberg and Isoholonomic Inequality", or section 3 of my "The Isoholonomic Problem and Some of its applications" which you can get from my web page, under publications. In these papers I get language and inspiration from quantum mechanics.

The arena of quantum mechanics is built out of Hilbert spaces. Write $\mathcal{H}$ for a fixed Hilbert space and $\langle \cdot, \cdot \rangle$ for its Hermitian inner product. Out of $\mathcal{H}$ we can build :

$S(\mathcal{H})$ = sphere in $\mathcal{H} = \{\psi \in \mathcal{H} : \langle \psi, \psi \rangle = 1\} \subset \mathcal{H}$. Its elements are called "normalized states".

$I\!P(\mathcal{H})$ = space of complex one-dimensional subspace of $\mathcal{H}$ is the projective space of $\mathcal{H}$.

It is time to throw Lie groups into the mix:

$U(\mathcal{H})$ = the unitary group of $\mathcal{H}$, which is the group of all complex linear transformations of $\mathcal{H}$ preserving the Hermitian inner product; $\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle$ for all $\psi, \phi \in \mathcal{H}$. The symmetry group of quantum mechanics is the unitary group.

Let us now suppose that $dim_{\mathbf{C}}\mathcal{H} = n$. Then $\mathcal{H}$ admits an Hermitian orthonormal basis $e_1, \ldots, e_n$ which yields and identification $\mathcal{H} \cong \mathbf{C^n}$ under which the Hermitian inner product turns to the standard one on $\mathbf{C^n}$.

Convention. The Hermitian inner product is complex linear in the second slot, and complex anti-linear in the second: $\langle \sum v^i e_i, \sum z^j e_j \rangle = \sum \bar{v}^i z^i$.

Exercise 1 *Show that in matrix terms, using a Hermitian orthonormal basis that the unitary group is defined by the quadratic equations $UU^* = Id$ where U^* is the conjugate transpose of $U \in M_n(\mathbf{C})$, where $M_n(\mathbf{C})$ denotes the vector space of all complex n by n matrices and where $Id \in M_n(\mathbf{C})$ is the identity matrix. Write $sym(n)$ for the space of all self-adjoint matrices.*

a) Show that the map $F : M_n(\mathbf{C}) \to \mathbf{sym(n)}$ is a polynomial map between real vector spaces and that Id is a regular value for this mapping.

b) Use basic manifold theory to conclude that the unitary group is a submanifold of $M_n(\mathbf{C})$.

c) Identify the tangent space to $U(n)$ at the identiy as the kernel of dF at Id. Describe this kernel in matrix terms. It is the lie algebra of the unitary group.

d) Compute the dimension of $U(n)$ and codimension of $U(n)$ within $M_n(\mathbf{C})$

e) Show that $U(n)$ is compact.

f) Show that $U(n)$ is connected.

e) Show that group multiplication and inversion are smooth maps on $U(n)$

Definition 1 *Any manifold G satisfying property (e) is called a Lie group.*

Remark. A basic theorem asserts that the **closed subgroups** of the group $GL(n, I\!R)$ and $GL(n, \mathbf{C})$, these being the groups of all real and complex invertible n by n matrices, are Lie groups. These are called "matrix Lie groups". You have proved above, by hand, that $U(n) \subset Gl(n, \mathbf{C})$ is a Lie group.

Definition 2 *The Hopf fibration is the circle fibration* $\pi : S(\mathcal{H}) \to I\!P(\mathcal{H})$ *which sends* ψ *to* $\pi(\psi) = [\psi] = span_{\mathbf{C}}\psi$.

The Hopf fibration is a $U(\mathcal{H})$ equivariant: $\pi(g\psi) = g\pi(\psi)$ where $g \in U(\mathcal{H})$ acts in the obvious way on $S(\mathcal{H})$ and on $I\!P(\mathcal{H})$.

1 Metric and Connection structure of Hopf fibration. Fubini-Study metric.

Use the standard induced metric on the sphere $S(\mathcal{H})$.

Definition 3 *The standard connection for the Hopf fibration is the connection whose horizontal distribution* $H \subset TS(\mathcal{H})$ *is defined by declaring that* H_ψ *is the orthogonal complement to the fiber of* π *through* ψ, *for each* $\psi \in S(\mathcal{H})$, *with the orthogonal defined by the the standard metric on the sphere, which is the real part of the Hermitian inner product.*

Thus:

$$H_\psi = ker(d\pi_\psi)^{\perp_{I\!R}}.$$

where we have used "$\perp_{I\!R}$" for the perpindicular with respect to the real part of the metric– which is a real inner product, and which is to be contrasted with $\perp$, the Hermitian perpindicular.

Proposition 1

$$H_\psi = \psi^\perp \tag{1}$$

with corresponding connection form is given by

$$A(\psi, d\psi) := \langle \psi, d\psi \rangle \tag{2}$$

viewed as an $iI\!R$*-valued one-form,* $iI\!R$ *being the Lie algebra of* S^1.

Proof of eq (1).The Hermitian metric induces a real inner product by taking its real part, and the intrinsic metric on the sphere is this real inner product, restricted to the tangent spaces of the sphere. Let us write "$\perp_{I\!R}$ for the perpindicular with respect to the real metric, to be contrasted with $\perp$, the Hermitian perpindicular. We have that $T_\psi S(\mathcal{H}) = \psi^{\perp_{I\!R}}$. The fiber of π through ψ is the S^1 orbit $e^{it}\psi$ so its tangent space, $ker(d\pi_\psi)$ is spanned by $i\psi$. (NOTE: $\psi \perp_{I\!R} i\psi$!). Then $H_\psi = (i\psi)^{\perp_{I\!R}} \cap \psi^{\perp_{I\!R}}$. Now verify that for any vector ψ we have that $(i\psi)^{\perp_{I\!R}} \cap \psi^{\perp_{I\!R}} = \psi^\perp$.

Proof of eq (2). In formula (2) the form $d\psi$ is the $\mathcal{H}$-valued one form that sends $v \in T_\psi S(\mathcal{H})$ to v viewed as a vector in $\mathcal{H}$. It is the differential of the $\mathcal{H}$-valued inclusion of $S(\mathcal{H}) \to \mathcal{H}$. First note that $ker A(\psi) = \psi^\perp = H_\psi$ which says that A defines the desired horizontal space. Second, we check the normalization property. Above we saw that the infinitesimal generator $\frac{\partial}{\partial\theta}$ of the circle action is the vector field $\frac{\partial}{\partial\theta}(\psi) = i\psi$. Evaluation $A(\psi)(\frac{\partial}{\partial\theta}(\psi) = \langle \psi, i\psi \rangle = i$, which is the correct normalization, if we identify $I\!R$ with $iI\!R$ by the linear map $1 \mapsto i$.

2 curvature

We compute $dA = d\langle\psi, d\psi\rangle = \langle d\psi, d\psi\rangle$. This takes some parsing!. It does not look like a two-form. How is it a two form? We have argued that $d\psi(v) = v \in \mathcal{H}$ for $v \in T_\psi S(\mathcal{H})$. Then, in some sense it must be that the curvature $dA(v, w) = \langle v, w\rangle$.

Proposition 2 *The imaginary part of the Hermitian inner product is a two-form,indeed a symplectic form on $\mathcal{H}$, when $\mathcal{H}$ is viewed as a real vector space. The pull-back of this two-form to the sphere $S(\mathcal{H})$ is a two form on the sphere whose kernel is the Hopf direction $\frac{\partial}{\partial\theta}$. This two-form is dA, that is to say:*

$$dA(v, w) = Im(\langle v, w\rangle)$$

for $v, w \in v, w \in \langle\psi, d\psi\rangle$

Now, in coordinates??
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The tautological line bundle $\gamma \to IP(\mathcal{H})$ is the complex line bundle that attaches to the point $p \in IP(\mathcal{H})$, the complex one-dimensional subspace $\gamma_p = p$. In other words,

$$\gamma = \{p, v) : p \in IP(\mathcal{H}), v \in \mathcal{H}, v \in p\} \subset IP(\mathcal{H}) \times \mathcal{H}$$

Observation: The trivial vector bundle $IP(\mathcal{H}) \times \mathcal{H}$ is isomorphic to $\gamma \oplus \gamma^\perp$ where $\gamma_p^\perp = (\gamma_p)^\perp$ with the perp taken with respect to the Hermitian inner product.

Proof. Any $v \in \mathcal{H}$ can be decomposed into a part tangentail to, and orthogonal to $p = span(\psi)$.

For the tautological line bundle we have