# Manifolds I
# Midterm

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November 16, 2016

1. **Proof** The straightening lemma implies that there are coordinates $x^j$ such that $X = \frac{\partial}{\partial x^1}$. Putting $Y = f^j \frac{\partial}{\partial x^j}$ we have that

$$\begin{aligned}[X,Y] &= \frac{\partial}{\partial x^1}(f^j \frac{\partial}{\partial x^j}) - f^j \frac{\partial^2}{\partial x^j \partial x^1} \\ &= f^j \frac{\partial^2}{\partial x^j \partial x^1} + \frac{\partial f^j}{\partial x^1}\frac{\partial}{\partial x^j} - f^j \frac{\partial^2}{\partial x^j \partial x^1} \\ &= \frac{\partial f^j}{\partial x^1}\frac{\partial}{\partial x^j}\end{aligned}$$

and setting this equal to $X = \frac{\partial}{\partial x^1}$ we obtain $f^j \frac{\partial}{\partial x^j} = \delta_{1,j}$ so that $f^1 = x^1$ and $f^{j \neq 1} = 0$ is a solution. Differentiating and substitution into the expression above confirms this result. ∎

2. (i)

   (ii)

3. (i) **Proof** Let $\phi_x$ be the coordinate chart $[x,y] = [\frac{x}{y},1] \mapsto \frac{x}{y} =: x$, that is $[x,1] \mapsto x$. Then we have $g.x = \phi_x^{-1}(g.[x,1]) = \phi_x^{-1}([ax+b, cx+d]) = \phi_x^{-1}([\frac{ax+b}{cx+d},1]) = \frac{ax+b}{cx+d}$

   (ii) First note that $E_{jj}^2 = E_{jj}$ for $j = 1,2$ and $E_{i,j}^2 = E_{ji}^2 = 0$ for $i = 1, j = 2$. The first relation implies that $\frac{d}{dt}|_{t=0}(e^{tE_{jj}}.[x,1]) = \frac{d}{dt}|_{t=0}(1 + (e^t - 1)E_{jj}).[x,1]$ and the second relation implies that $\frac{d}{dt}|_{t=0}(e^{tE_{\alpha,\beta}}.[x,1]) = \frac{d}{dt}|_{t=0}(1 + tE_{\alpha,\beta}).[x,1]$. From this we obtain

$$\begin{aligned}\frac{d}{dt}|_{t=0}(e^{tE_{11}}.[x,1]) &= \frac{d}{dt}|_{t=0}(1 + (e^t - 1)E_{11}).[x,1] \\ &= \frac{d}{dt}|_{t=0}([xe^t,1]) \\ &= \frac{d}{dt}|_{t=0}(xe^t) \\ &= xe^t|_{t=0} \\ &= x\end{aligned}$$

   Hence $f_{11}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x}$ and

$$\begin{aligned}\frac{d}{dt}|_{t=0}(e^{tE_{22}}.[x,1]) &= \frac{d}{dt}|_{t=0}([x,e^t]) \\ &= \frac{d}{dt}|_{t=0}(xe^{-t}) \\ &= -xe^{-t}|_{t=0} \\ &= -x\end{aligned}$$

So that $f_{22}\frac{\partial}{\partial x} = -x\frac{\partial}{\partial x}$. And the second relation yields

$$\begin{aligned}\frac{d}{dt}|_{t=0}(e^{tE_{12}}.[x,1]) &= \frac{d}{dt}|_{t=0}(1+tE_{11}).[x,1] \\ &= \frac{d}{dt}|_{t=0}(x+t,1) \\ &= \frac{d}{dt}|_{t=0}(x+t) \\ &= 1|_{t=0} \\ &= 1\end{aligned}$$

So that $f_{12}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$ and, finally,

$$\begin{aligned}\frac{d}{dt}|_{t=0}(e^{tE_{21}}.[x,1]) &= \frac{d}{dt}|_{t=0}(1+tE_{21}).[x,1] \\ &= \frac{d}{dt}|_{t=0}(x,1+tx) \\ &= \frac{d}{dt}|_{t=0}(\frac{x}{1+tx}) \\ &= \frac{-x^2}{(1+tx)^2}|_{t=0} \\ &= -x^2\end{aligned}$$

So that $f_{2,1}\frac{\partial}{\partial x} = -x^2\frac{\partial}{\partial x}$

(iii) It is not surprising that $\sigma(E_{11}) + \sigma(E_22) = 0$ because we have

$$\begin{aligned}\frac{d}{dt}|_{t=0}(e^{t(E_{11}+E_{22})}[x,1]) &= \frac{d}{dt}|_{t=0}(e^{tI}[x,1]) \\ &= \frac{d}{dt}|_{t=0}(e^tI[x,1]) \\ &= \frac{d}{dt}|_{t=0}(e^t[x,1]) \\ &= \frac{d}{dt}|_{t=0}([e^tx,e^t]) \\ &= \frac{d}{dt}|_{t=0}(\frac{e^tx}{e^t}) \\ &= \frac{d}{dt}|_{t=0}(x) \\ &= 0 \quad \blacksquare\end{aligned}$$

(iv) We have been using the coordinate chart $[x, y] = [x/y, 1] \mapsto x/y =: x$, i.e. $[x, 1] \mapsto x$. Under this coordinate chart, the solution to $\frac{dx}{dt} = -x^2$ is $x = \frac{-1}{t+c}$. This appears to be singular at $t = -c$. However, consider the effect of transitioning to the chart $[x, y] = [1, y/x] \mapsto y/x =: y$, i.e. $[1, y] \mapsto y$. If the first coordinate chart is $\phi$ and the second $\psi$, we have the transition map $\psi \circ \phi^{-1}(x) = \frac{1}{x}$. Hence, under this coordinate transformation the solution becomes $x = t+c$ and this is no longer singular at $t = -c$. It seems that the apparent singularity was merely an artifact of the coordinate chart.