Manifolds I
Midterm Makeup Problem 2

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1. Show that $df(p) = 0$ implies $\frac{\partial^2 f}{\partial x^i \partial x^j}|_p dx^i dx^j = \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}|p dy^i dy^j$

   Proof Note that $dx^i dx^j = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}|_p dy^\alpha dy^\beta$ (using summation convention) and that

$$\begin{aligned}\frac{\partial^2 f}{\partial x^i \partial x^j} &= \frac{\partial}{\partial x^i}\frac{\partial f}{\partial x^j} \\ &= \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial x^j}\frac{\partial f}{\partial y^\alpha}\right) \\ &= \frac{\partial y^\alpha}{\partial x^j}\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial y^\alpha}\right) + \frac{\partial f}{\partial y^\alpha}\frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial x^j}\right) \\ &= \frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 f}{\partial y^\beta \partial y^\alpha} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\end{aligned}$$

   so that $\frac{\partial^2 f}{\partial x^i \partial x^j}|_p = \left(\frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 f}{\partial y^\beta \partial y^\alpha} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\right)|_p$ hence

$$\begin{aligned}\frac{\partial^2 f}{\partial x^i \partial x^j}|_p dx^i dx^j &= \left(\frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 f}{\partial y^\beta \partial y^\alpha} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\right)|_p\left(\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}|_p\right) dy^\alpha dy^\beta \\ &= \left(\frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial^2 f}{\partial y^\beta \partial y^\alpha}\right)|_p dy^\alpha dy^\beta + \left(\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\frac{\partial f}{\partial y^\alpha}\right)|_p dy^\alpha dy^\beta \\ &= \frac{\partial^2 f}{\partial y^\beta \partial y^\alpha}|_p dy^\alpha dy^\beta + \left(\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\frac{\partial f}{\partial y^\alpha}\right)|_p dy^\alpha dy^\beta\end{aligned}$$

   so that if $\frac{\partial f}{\partial x^i}|_p dx_i = 0$ (i.e. $\frac{\partial f}{\partial x^i} = 0$ for each i, then $\frac{\partial y^\alpha}{\partial x^i}\frac{\partial f}{\partial y^\alpha}|_p = 0$ for each i, hence $\left(\frac{\partial}{\partial x^j}\frac{\partial y^\alpha}{\partial x^i}\right)|_p = \left(\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i}\right)|_p = 0$ and, consequently $\left(\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\frac{\partial f}{\partial y^\alpha}\right)|_p dy^\alpha dy^\beta = 0$ so that, in fact $\frac{\partial^2 f}{\partial x^i \partial x^j}|_p dx^i dx^j = \frac{\partial^2 f}{\partial y^\beta \partial y^\alpha}|_p dy^\alpha dy^\beta$ ∎

2. According to the normal form or straightening lemma there are coordinates x^i centered at p such that $f = f(p) + x^1$. That is, the gradient of f is just $df = dx^1$

3. In these normal form coordinates, we have $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^i \partial x^j}(x^1 + f(p)) = \frac{\partial}{\partial x^i}(\delta^{1,j}) = 0$ so that the Hessian is zero.

4. If we consider the gradient $X_p = \frac{\partial f}{\partial x^i}|_p dx^i$ to be a function of p, then it defines a vector field and we can apply the transformation used in problem 2 of the midterm in order to obtain a coordinate transformation making the jacobian of X_p have the form M^i_j. However, instead of writing

$$y^k = x^k + \frac{1}{2}a^k_1(x^1)^2 + a^k_2 x^1 x^k + \dots$$

we replace the y^k with $\frac{\partial y^k}{\partial x^1}$ and t hen integrate to find y^k. This will yield a quadratic in the x^k, except for the first term which will contain a cubic in x^1.