The 'arbitrary Hessian' problem

To show: if $df(p) \neq 0$.
then by a change of coordinates
we can insure

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(p)$$

is any symmetric matrix.

Starting point. Normal form for functions: $\exists$ coordinates $(x^1, \dots, x^n)$ centered at p $(x^i(p) = 0)$ s.t. in these coordinates

$$f(x^1, \dots, x^n) = x^1 + f(p)$$

$n=1$. WLOG $p = 0$, $f(x) = x + C$.

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial^2 f}{\partial x^2} = 0.$$

Change of coordinates

$$y = x + ax^2$$

What is $\left.\frac{\partial^2 f}{\partial y^2}\right|_0$?

Way 1 $\quad x = y - ax^2 = y - ay^2 + O(y^3)$

But $\quad f(x) = x + C$

So in terms of y:

$$f = y - ay^2 + O(y^3)$$

$$\frac{\partial f}{\partial y} = -2ay + O(y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = -2a.$$

$n=1$. Way 2.

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}\frac{dx}{dy}$$

But

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial x}{\partial y} = \frac{1}{\frac{dy}{dx}}$$

& $$\frac{dy}{dx} = \frac{d}{dx}(x+ax^2) = 1+2ax$$

So $$\frac{dx}{dy} = \frac{1}{1+2ax} = 1-2ax+O(x^2)$$

& $$\frac{\partial f}{\partial y} = 1\cdot(1-2ax)+O(x^2)$$

& $$\left.\frac{\partial^2 f}{\partial y^2}\right|_{y=0} = -2a.$$

✓

checks with computation, way 1.

General n

Again: $f = x^1 + C$ in x-coordinates

Try

$$y^1 = x^1 + \sum A_{ij} x^i x^j$$
$$y^2 = x^2$$
$$y^3 = x^3$$
$$\vdots$$
$$y^n = x^n.$$

arbitrary symmetric

<u>Way 1</u>. ~~*~~ invert coordinate change

$$x^1 = y^1 - \sum A_{ij} x^i x^j$$

But $x^i = y^i + O(y^2)$

(for $i > 1$, $x^i = y^i$, for $i = 1$, $x^1 = y^1 + O(y^2)$)

so

$$x^1 = y^1 - \sum_{\text{all } i,j} A_{ij} y^i y^j + O(y^3).$$

$$f = x^1 + C = y^1 - \sum A_{ij} y^i y^j + C + O(y^3)$$

$$\frac{\partial f}{\partial y^i} = \delta^1_i - 2 A_{ij} y^j + O(y^2)$$

$$\frac{\partial^2 f}{\partial y^j \partial y^i} = -2 A_{ij} + O(y)$$

so at 0:

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = -2 A_{ij} = \text{arbitrary symmetric.}$$

Way 2

$$\frac{\partial f}{\partial y^i} = \frac{\partial f}{\partial x^l}\frac{\partial x^l}{\partial y^i}$$

But $\frac{\partial f}{\partial x^l} = \begin{cases} 1, & l = 1 \\ 0, & l \neq 1. \end{cases}$

So

$$\frac{\partial f}{\partial y^i} = \frac{\partial x^1}{\partial y^i}$$

Is the $1, i$ entry of the Jacobian of the Inverse transformation of $\vec{x} \longrightarrow \vec{y}(\vec{x})$.

Now $\frac{\partial \vec{x}}{\partial \vec{y}} = \left(\frac{\partial \vec{y}}{\partial \vec{x}}\right)^{-1}$

$$\begin{aligned} y^1 &= x^1 + \Sigma A_{ij} x^i x^j \\ y^2 &= x^2 + \\ &\vdots \\ y^n &= x^n \end{aligned}$$

$$\frac{\partial \vec{y}}{\partial \vec{x}} = \begin{pmatrix} 1 + 2\Sigma A_{1j} x^j & 2\Sigma A_{2j} x^j, & \cdots & \Sigma A_{nj} x^j \\ 0 & 1 & 0 - - & 0 \\ 0 & 0 & 1 & 0 - - \\ 0 & & & \\ | & & & \\ 0 & 0 - & - & - 1 \end{pmatrix}$$

way 2 ctd

Or

$$\frac{\partial \vec{y}}{\partial \vec{x}} = Id + 2\vec{h}$$

where $h = h(x)$ is the rank 1 matrix whose first row is

$$\vec{h}_1(x) = (2\sum A_{1j}x^j, 2\sum A_{2j}x^j, \ldots, 2\sum A_{nj}x^j)$$

Now as $\vec{x} \to 0$, $\vec{h} \to 0$
& for h small

$$(I + h)^{-1} = I - h + O(h^2)$$

which for $n \times n$ matrices
(or any $\mathbb{R}$-algebra!)

Thus: $\frac{\partial \vec{x}}{\partial \vec{y}} \approx Id - \cancel{2}h(x) + O(h(x))^2$.

& since $\vec{x} = \vec{y} + O(x^2) = \vec{y} + O(y^2)$
we have

$$\frac{\partial \vec{x}}{\partial \vec{y}} = Id - h(y)$$

or $\frac{\partial x^i}{\partial y^j} = \delta^i_j - 2\sum A_{jk} y^k$

way 2 ctd.

Finally since $\frac{\partial f}{\partial y^j} = \frac{\partial x^1}{\partial y^j}$

we have

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = \frac{\partial}{\partial y^i}\left(\delta^1_j - 2\sum A_{jk} y^k + O(y^2)\right)$$

$$= -2A_{ij} + O(y).$$

& at 0:

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = -2A_{ij}.$$

= arbitrary symmetric
since A_{ij} is
arbitrary, symmetric.