TANGENT VECTORS. THREE OR FOUR DEFINITIONS.

RMONT

We define and try to understand the tangent space of a manifold Q at a point q, as well as vector fields on a manifold. The tangent space at $q \in Q$ is a real vector space intrinsically attached to the manifold. It is denoted by T_pqQ. Its dimension is n, the dimension of the manifold. If $x^i, i = 1, \dots, n$ are coordinates in a nbhd of q then the expressions $\frac{\partial}{\partial x^i}$ form a basis for T_qQ. If $F : Q \to M$ is a smooth map, then its differential $dF_q : T_qQ \to T_{F(q)}M$ is a linear map between tangent spaces.

Three definitions. For Q an abstract manifold, not embedded in any Euclidean space, there are three equivalent definition of the tangent space at $q \in Q$.

1) as derivations acting on functions defined near q.

2) as equivalence classes of curves passing through q

3) an operational definition, directly from coordinates.

We will need to understand all three definitions, and how to go back and forth between them.

If $Q \subset \mathbb{R}^N$ is embedded, then there is a 4th definition,

4) If L is an n-dimensional linear subspace of $\mathbb{R}^N$, then $q + L$ is an affine space passing through q. Among all n-dimensional linear subspaces $L \subset \mathbb{R}^N$, the tangent space T_qQ is that linear subspace such that $q + T_qQ$ is the best approximation to Q in the usual calculus sense.

1. Vector fields as Derivations.

From beginning calculus we know that the operation $f \mapsto Df := df/dx$ is a linear operator on the algebra $A = C^\infty(\mathbb{R})$. In addition to being linear, it satisfies the Liebnitz identity:

$$D(fg) = fDg + gDf, \text{ for all } f, g \in A \tag{1}$$

Definition 1. *Let $\mathbb{K}$ be a field and A a $\mathbb{K}$ algebra with unit. Then a derivation on A is a $\mathbb{K}$-linear operator $D : A \to A$ satisfying the Liebnitz identity , eq (1).*

We can now give a definition of ‘vector field on a manifold”.

Definition 2. *Let $A = \mathbb{C}^\infty(Q)$ be the algebra of smooth functions on a manifold Q. Then a vector field is a derivation of A. The space of all vector fields forms is denoted by $\chi(Q)$ or by $\Gamma(TQ)$.*

Exercise 1. *Prove that if $D : A \to A$ is a derivation of A, then $D(1) = 0$ where 1 is the unit of the commutative $\mathbb{K}$-algebra A*

Example 1. The partial derivatives $\frac{\partial}{\partial x^i}$ are derivations of $A = \mathbb{C}^\infty(\mathbb{R}^n)$.

Exercise 2. *Prove that If $D : A \to A$ is a derivation of A, and if $h \in A$ then hD is also a derivation, where $(hD)(g) = h(Dg)$. Show that the vector space $Der(A)$ of all derivations of A is a module over A.*

It follows from the previous example and the exercise above that

$$X = \Sigma_{i=1}^{n} X^i \frac{\partial}{\partial x^i}; X^i \text{ smooth functions on } \mathbb{R}^n \tag{2}$$

are derivations of $\mathbb{R}^n$.

Theorem 1. *Any vector field on $\mathbb{R}^n$ can uniquely be expressed in the form of eq (2) where the X^i are smooth functions on $\mathbb{R}^n$*

2. Tangent Space. Cotangent space, algebraic definitions.

If X is a smooth vector field on M, and $q \in M$ then $X(q)$ should be a 'vector' attached at q. The vector space to which it is attached is denoted T_qM and called the tangent space at q. Let us set

$$v = X(q) \in T_qM$$

and agree that the meaning of v is as a linear map $v : \mathbb{C}^\infty(M) \to \mathbb{R}$ defined by $v[f] = X[f](q)$. From $X[fg] = fX[g] + gX[f]$ we see that

$$v[fg] = f(q)v[g] + g(q)v[f] \tag{3}$$

This suggests the definition:

Definition 3. *The tangent space at $q \in M$, henceforth denoted by T_qQ, is the space of linear functionals $C^\infty(M) \to \mathbb{R}$ satisfying the additional derivation condition of eq (3).*

Now please observe that if $f, g \in \mathfrak{m}_p$, the ideal of functions vanishing at q, then $v[fg] = 0$. The linear span of functions of this form $fg, f, g \in \mathfrak{m}_q$ is denoted $\mathfrak{m}_p^2$ and is a subalgebra: $\mathfrak{m}_q^2 \subset \mathfrak{m}_q \subset C^\infty(M)$.. By linearity v vanishes on $\mathfrak{m}_q^2$. Furthermore, note that if $f \in C^\infty(M)$ then $f - f(q) = f - f(q)1 \in \mathfrak{m}_q$ and that $v[f] = v[f - f(q)]$ since $v[1] = 0$. We have proved the first half of :

Proposition 1. *Each $v \in T_qM$ induces a linear map $\mathfrak{m}_q/\mathfrak{m}_q^2 \to \mathbb{R}$, and this linear functional uniquely determines v. Moreover, every such linear map is induced by a derivation $v \in T_qM$.*

The first half of the proposition asserts the existence of a canonical linear injection $T_qM \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$. The second, as yet unproved, half of the proposition asserts that this injection is onto, and hence an isomorphism.

INJECTIVITY. Our next goal is to prove this second half, that is, to show that the linear injection of the proposition is indeed onto. With this in mind, we introduce another basic object of manifold theory.

Definition 4. *The cotangent space at T_q^*M at q is $\mathfrak{m}_q/\mathfrak{m}_q^2$.*

Exercise 3. , *Show that $T_0^*\mathbb{R}^n$ is an n-dimensional vector space whose basis is the linear coordinate functions x^i, taken mod the ideal generated by their products $x^i x^j$.*

Solution If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth near zero and if $f(0, \dots, 0) = 0$ then Taylor tells us that

$$f(x^1, x^2, \dots, x^n) = \Sigma a_i x^i + \Sigma b_{ij} x^i x^j + O(x^3).$$

Thus $f \equiv \Sigma a_i x^i (mod \mathfrak{m}_0^2)$ and the x^i mod $\mathfrak{m}_0^2$ form a basis for T_p^*M.

As a consequence of the exercise we find:

Proposition 2. *$T^*_q M$ is a real vector space of dimension $n = dim(M)$. Coordinates $x^1, \ldots, x^n$ defined in a nbhd of q induce a basis denoted $dx^1, \ldots dx^n$ and defined by eq (4) below.*

Proof. Let $x^1, \ldots, x^n$ be coordinates defined via chart whose domain contains a nbhd of q. Subtract the constants $c^i = x^i(p)$ from the x^i to get that $x^i - c^i \in \mathfrak{m}_q$. Then

$$dx^i := x^i - x^i(q)(mod \mathfrak{m}_q^2) \tag{4}$$

define elements of $T^*_q Q = \mathfrak{m}_q/\mathfrak{m}_q^2$. Since the coordinate chart yields a diffeomorphism between the nbhd of q and $\mathbb{R}^n$, and since the coordinate functions form a basis for $T^*_0 \mathbb{R}^n$, these dx^i form a basis for $T^*_q Q$. QED

COMMENTARY on PROOF SHORTCOMINGS. It may bother you, rightly so, that in the proof above the x^i are actually not functions on all of Q. There are two ways to deal with this shortcoming.

- (1) Extend the x^i to all of M by 'bumping them off" outside a smaller nbhd of q by using a bump function β, so they are extended outside of a nbhd of q to be zero.
- (2): Go back to the definitions of $\mathfrak{m}_q$ etc and make everything local by doing things on the level of *germs*, these being at heart maps or functions defined in arbitrarily small nbhds of a point.

Basis dx^i. Dual basis $\frac{\partial}{\partial x^i}$.

The dual basis to the basis dx^i for $T^*_q M$ is written $\frac{\partial}{\partial x^i}$. Thus:

$$dx^i(\frac{\partial}{\partial x^j}) := \frac{\partial x^i}{\partial x^j} = \delta^i_j$$

Using this relation, linearity, the Taylor expansion as per above, we see that $\frac{\partial}{\partial x^i}$ acts on $C^\infty(M)$ by expressing $f \in C^\infty(M)$ in the local coordinates x^i and taking the resulting partial derivative, and evaluating appropriately:

$$\frac{\partial}{\partial x^i}|_q[f] = \frac{\partial f \circ \phi}{\partial x^i}|_{\phi^{-1}(q)}.$$

where $\phi : \mathbb{R}^n \dashrightarrow M$ is the inverse of the coordinate chart whose components are $(x^1, \ldots, x^n)$.

This yields the alternative definition

Definition 5. *The tangent space $T_q M$ at q is the dual vector space to the cotangent space. Thus: $T_q M = (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$.*

3. Curves as tangent vectors. Language of germs.

The most "geometric" way to define tangent vectors at q is as equivalence classes of curves passing through q. We only need tiny arcs of such curves. Similarly, for defining the cotangent space, we only need functions defined in tiny nbhds of the point q in question. See item (1) on "COMMENTARY on Proof Shortcomings" above. With these two applications in mind, we take a few moments to define the notion of the "germ of a map".

Let X and Y be smooth manifolds and $p \in X$. Write $f : (X, p) \dashrightarrow Y$ to mean that f has domain some nbhd U of p and that $f : U \to Y$ is smooth. We may also fix a point $q \in Y$ and then we write $f : (X, p) \dashrightarrow (Y, q)$ to mean that we also impose the condition that $f(p) = q$.

Definition 6. *Two smooth maps $f, g : (X, p) \dashrightarrow Y$ have the same germ if there is a nbhd V of p on which both f and g are defined and such that such that $f|_V = g|_V$.*

Exercise 4. *The space of all germs of functions $(M, p) \to \mathbb{R}$ forms a local ring, denoted $C^\infty(M)_p$ whose unique maximal ideal consists of the germs of functions vanishing at p which we will continue to write as $\mathfrak{m}_p$.*

Exercise 5. *If $f, g : (\mathbb{R}^n, 0) \to \mathbb{R}$ have the same germ, then their Taylor expansions agree to all orders.*

Definition 7. *Two smooth functions $F, G : (\mathbb{R}^n, 0) \to (\mathbb{R}^d, 0)$ agree to first order if $|F(x) - G(x)| = O(|x|^2)$*

Basic calculus. Two smooth $F, G : (\mathbb{R}^n, 0) \to (\mathbb{R}^d, 0)$ agree to first order iff $DF(0) = DG(0)$ as linear maps $\mathbb{R}^n \to \mathbb{R}^d$.

Definition 8. *Two smooth maps $f, g : (X, p) \dashrightarrow (Y, q)$ "agree to first order" if in some coordinate system $\psi_X : \mathbb{R}^n \dashrightarrow X$ and $\phi_Y : \mathbb{R}^d \dashrightarrow Y$ with nbhds containing p, q, we have that $\phi_Y^{-1} \circ f \circ \phi_X$ and $\phi_Y^{-1} \circ g \circ \phi_X$ agree to first order as germs $(\mathbb{R}^n, \psi_X^{-1}(p)) \to (\mathbb{R}^d, 0, \psi_Y^{-1}(q))$.*

Exercise 6. *If two smooth maps f, g as above agree to first order rel. one pair of coordinate systems, then they agree to first order rel. any other compatible pair of coordinates systems.*

The relation "agree to first order" is a well defined equivalence relation and we can take the resulting space of equivalence classes.

Definition 9. *Let $c : (\mathbb{R}, 0) \to (Q, q)$ be a curve. Write $c'(0)$ or $dc/dt(0)$ for the equivalence class consisting of all curve germs $(\mathbb{R}, 0) \to (Q, q)$ which agree to first order with q.*

Definition 10. *An element of the tangent space at $q \in Q$ is an equivalence class of curve germs $(\mathbb{R}, 0) \to (Q, q)$ where the equivalence relation is "agree to first order".*

Now the curve germ $c : (\mathbb{R}, 0) \to (Q, q)$ defines a derivation $v : C^\infty(M) \to \mathbb{R}$ by differentiation along the curve:

$$f \mapsto \frac{d}{dt}(f \circ c)|_{t=0} := v[f] \tag{5}$$

Exercise 7. *Verify that if two curve germs through q agree to first order at q then the derivations they define via eq (5) are equal.*

A coordinate computation now shows that every element of T_qQ, viewed as the dual space to $\mathfrak{m}_q/\mathfrak{m}_q^2$, arises as directional derivative along a curve.

The facts just described yield the curve definition of the tangent space. We could define the tangent space at q as the space whose elements are first order equivalence classes of curves passing through q. Equation 5 relates the curve and the derivation def'n. In words: tangent vectors ARE directional derivatives.

BASES. Under this isomorphism between the derivation definiton and the curve definition of the tangent space, the basis elements $\frac{\partial}{\partial x^i}$ of the derivation definition correspond to the tangent vectors of the "ith coordinate curves" $c_i(t) = \psi^{-1}(P + te_i) = \psi(0, \ldots, 0, t, 0, \ldots, 0)$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the usual standard basis of $\mathbb{R}^n$, and where $\psi : (Q, q) \to (\mathbb{R}^n, P)$ is our coordinate chart with coordinates x^i.

SUMMARY: There are TWO WAYS to think of tangent vectors, so far, (1) as derivations on functions or (2) as equivalence classes of curves. The space of all tangent vectors at a point to an n-manifold at a point p is an n dimensional real vector space denoted T_pM, with basis $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$, where x^i are coordinates at p. Composition of a function f with a curve c realizes the canonical duality pairing $T_p^*M \times T_pM \to \mathbb{R}$:

$$\frac{d}{dt}|_{t=0} f \circ c = df(p)(v) = v[f], \text{ where } df(p) \in T_p^*M, v = c'(0) \in T_pM.$$

4. Operational definition of Tangent space.

Instead of trying to say what a tangent vector is, we only say how it transforms when we change from one coordinate chart to another. This is the most useful definition, computationally. It has the disadvantage of giving us no picture of what a tangent vector actually is.

We know what a vector in $\mathbb{R}^n$ is. Viewed rel. a chart $\phi : M \dashrightarrow \mathbb{R}^n$ a tangent vector at p is a vector $V_\phi \in \mathbb{R}^n$ that we think of as being attached to the point $\phi(p) \in \mathbb{R}^n$. If $\psi : M \dashrightarrow \mathbb{R}^n$ is another chart containing p, then in that chart, the same vector will be represented by a different vector V_ψ, now attached at $\psi(p)$. The two vectors are related by the differential of the transition function between the two coordinates:

$$V_\psi = d(\psi \circ \phi^{-1})_{\phi(p)} V_\psi \tag{6}$$

If $\phi = (x^1, \ldots, x^n)$ then $\frac{\partial}{\partial x^i}$ is the notation we reserve for the vector in T_qQ whose representative in the ϕ-chart is the standard basis vector e_i, consisting of all zeros except a 1 in the ith place. What does this same vector look like in the ψ chart whose coordinates are $(y^1, y^2, \ldots, y^n)$? According to eq (6), we have that e_i in the ϕ-chart corresponds to the vector $d(\psi \circ \phi^{-1})_{\phi(p)} e_i$ in the ψ-chart. Now e_j in the ψ chart corresponds to the vector $\frac{\partial}{\partial y_j}$ in T_qM. Thus, the transformation formula yields the relation

$$\frac{\partial}{\partial x^i} = \Sigma_j \frac{\partial y^j}{\partial x^i}|_{\phi(p)} \frac{\partial}{\partial y^j}$$

To see this, note that $\psi \circ \phi^{-1}$ has the coordinate expression $y^i = y^i(x^1, \ldots, x^n)$.

Commentary. This last definition is the most computationally useful, but hides the geometry.

5. Normal forms lemmas

Lemma 1. *Let $f : M \dashrightarrow \mathbb{R}$ be a smooth function defined in a nbhd of $p \in M$ and suppose that $df(p) \neq 0$. Then there exist coordinates $x^1, \ldots, x^n$ about p such that in this coordinate nbhd*

$$f = x^1$$

SUMMARY OF THIS LEMMA: "Away from critical points, , all functions are locally the same"

Lemma 2. *Let $v \in T_pM$ with $v \neq 0$. Then there exist coordinates $x^1, \dots, x^n$ about p such that*

$$v = \frac{\partial}{\partial x^1}|_p.$$

Lemma 3 (Straightening Lemma. Sometimes called flowbox lemma)**.** *Let X be a vector field defined in a nbhd of $p \in M$ which does not vanish at p. Then there exist coordinates $x^1, \dots, x^n$ about p such that in this coordinate nbhd*

$$X = \frac{\partial}{\partial x^1}$$

SUMMARY OF STRAIGHTENING LEMMA: "Away from zeros, all vector fields are locally the same"

Proof of Lemma 1. Let $y^1, \dots y^n$ be arbitrary smooth coordinates about p. Then, since $df(p) = \Sigma \frac{\partial f}{\partial y^i} dy^i$ we have that at least one of the partial derivatives $\frac{\partial f}{\partial y^i}|p \neq 0$. Relabel this coordinate index if necessary so that it is $i = 1$ by switching the index i and 1 in the case that $\frac{\partial f}{\partial y^1}|_p = 0$. Define a map $\mathbb{R}^n \dashrightarrow \mathbb{R}^n$ by $(y_1, y_2, \dots, y_n) \mapsto (f(y^1, y^2, \dots, y^n), y^2, y^3, \dots, y^n)(x^1, x^2, \dots, x^n)$. Then the Jacobian of this transformation is

$$\begin{pmatrix} \frac{\partial f}{\partial y^1} & \frac{\partial f}{\partial y^2} & \frac{\partial f}{\partial y^2} & \frac{\partial f}{\partial y^2} & \cdots & \frac{\partial f}{\partial y^n} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

with determinant $\frac{\partial f}{\partial y^1}$. Hence this map is a local diffeo $\mathbb{R}^n \to \mathbb{R}^n$ near $\phi(p)$. Composing ϕ with this coordinate change yields coordinates $(x^1, x^2, \dots, x^n)$ for which $f = x^1$.

QED, lemma 1.

Pf of Lemma 2. Choose any chart $\phi = (u^1, u^2, \dots, u^n)$ about p. Since $v \neq 0$, in this chart v is represented by some nonzero vector $v_\phi \in \mathbb{R}^n$. Let A be an invertible linear tranformation taking v_ϕ to e_1. Then $\psi = A \circ \phi$ is a new chart, and the ψ to ϕ transition map is A. Thus in the ψ chart we have that $v_\psi = Av_\phi = e_1$, which means in that if $\psi = (x^1, x^2, \dots, x^n)$ that $v = \frac{\partial}{\partial x^1}$.

Proof of Straightening lemma. Since $X(p) \neq 0$, by lemma 2 we can find coordinates $u^1, u^2, \dots$ about p such tht $X(p) = \frac{\partial}{\partial u^1}$. Now consider the (local) hypersurface $u^1 = 0$ in M which is coordinatized by $u^2, \dots, u^n$ according to $(u^2, u^2, \dots, u^n) = \phi^{-1}(0, u^2, u^3, \dots, u^n)$. Let $\Phi_t : M \dashrightarrow M$ denote the flow of X. Define $F : \mathbb{R}^n \to M$ by $F(x^1, x^2, \dots, x^n) = \Phi_{x^1} \phi^{-1}(0, x^2, x^3, \dots, x^n)$. I claim that dF_0 is invertible , and hence by the inverse function theorem has an inverse and so F^{-1} are good coordinates. The x^i are our desired coordinates.

To show dF_0 is invertible we compute from the definitions. $dF_0(e_1) = \frac{d}{dt}|_{t=0} \Phi_t(\phi^{-1}(0)) = X(p) = \frac{\partial}{\partial x^1}$. And for $i > 0$ we have $dF_0(e_i) = d(\Phi_0) d\phi_0^{-1}(e_i) = \frac{\partial}{\partial x^i}$. Thus dF_0 maps basis to basis and so is invertible and yields good coordinates.

Finally, since $\frac{d}{dt}\Phi_t(q)|_{t=0} = X(q)$ and $\frac{\partial}{\partial x^1} = dF_{(x_1, x_2, \dots, x_n)}(e_1) = \frac{d}{dh}|_{h=0} \Phi_{x_1 + h}(\phi^{-1}(0, x^2, \dots, x^n))$ we see that in the new coordinates $\frac{\partial}{\partial x^1} = X$. QED