THE STRAIGHTENING LEMMA AND OTHER NORMAL FORM LEMMAS.

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Lemma 1. *Let $f : M \dashrightarrow \mathbb{R}$ be a smooth function defined in a nbhd of $p \in M$ and suppose that $df(p) \neq 0$. Then there exist coordinates $x^1, \ldots, x^n$ about p such that in this coordinate nbhd*

$$f = x^1$$

SUMMARY OF THIS LEMMA: "Away from critical points, , all functions are locally the same"

Lemma 2. *Let $v \in T_pM$ with $v \neq 0$. Then there exist coordinates $x^1, \ldots, x^n$ about p such that*

$$v = \frac{\partial}{\partial x^1}|_p.$$

Lemma 3 (Straightening Lemma. Sometimes called flowbox lemma)**.** *Let X be a vector field defined in a nbhd of $p \in M$ which does not vanish at p. Then there exist coordinates $x^1, \ldots, x^n$ about p such that in this coordinate nbhd*

$$X = \frac{\partial}{\partial x^1}$$

SUMMARY OF STRAIGHTENING LEMMA: "Away from zeros, all vector fields are locally the same"

Proof of Lemma 1. Let $y^1, \ldots y^n$ be arbitrary smooth coordinates about p. Then, since $df(p) = \Sigma \frac{\partial f}{\partial y^i} dy^i$ we have that at least one of the partial derivatives $\frac{\partial f}{\partial y^i}|p \neq 0$. Relabel this coordinate index if necessary so that it is $i = 1$ by switching the index i and 1 in the case that $\frac{\partial f}{\partial y^1}|_p = 0$. Define a map $\mathbb{R}^n \dashrightarrow \mathbb{R}^n$ by $(y_1, y_2, \ldots, y_n) \mapsto (f(y^1, y^2, \ldots, y^n), y^2, y^3, \ldots, y^n)(x^1, x^2, \ldots, x^n)$. Then the Jacobian of this transformation is

$$\begin{pmatrix} \frac{\partial f}{\partial y^1} & \frac{\partial f}{\partial y^2} & \frac{\partial f}{\partial y^2} & \frac{\partial f}{\partial y^2} & \cdots & \frac{\partial f}{\partial y^n} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

with determinant $\frac{\partial f}{\partial y^1}$. Hence this map is a local diffeo $\mathbb{R}^n \to \mathbb{R}^n$ near $\phi(p)$. Composing ϕ with this coordinate change yields coordinates $(x^1, x^2, \ldots, x^n)$ for which $f = x^1$.

QED, lemma 1.

Pf of Lemma 2. Choose any chart $\phi = (u^1, u^2, \ldots, u^n)$ about p. Since $v \neq 0$, in this chart v is represented by some nonzero vector $v_\phi \in \mathbb{R}^n$. Let A be an invertible linear tranformation taking v_ϕ to e_1. Then $\psi = A \circ \phi$ is a new chart, and the ψ to ϕ transition map is A. Thus in the ψ chart we have that $v_\psi = Av_\phi = e_1$, which means in that if $\psi = (x^1, x^2, \ldots, x^n)$ that $v = \frac{\partial}{\partial x^1}$.

Proof of Straightening lemma. Since $X(p) \neq 0$, by lemma 2 we can find coordinates $u^1, u^2, \ldots$ about p such tht $X(p) = \frac{\partial}{\partial u^1}$. Now consider the (local) hypersurface $u^1 = 0$ in M which is coordinatized by $u^2, \ldots, u^n$ according to $(u^2, u^2, \ldots, u^n) = \phi^{-1}(0, u^2, u^3, \ldots, u^n)$. Let $\Phi_t : M \dashrightarrow M$ denote the flow of X. Define $F : \mathbb{R}^n \to M$ by $F(x^1, x^2, \ldots, x^n) = \Phi_{x^1}\phi^{-1}(0, x^2, x^3, \ldots, x^n)$. I claim that dF_0 is invertible , and hence by the inverse function theorem has an inverse and so F^{-1} are good coordinates. The x^i are our desired coordinates.

To show dF_0 is invertible we compute from the definitions. $dF_0(e_1) = \frac{d}{dt}|_{t=0}\Phi_t(\phi^{-1}(0)) = X(p) = \frac{\partial}{\partial x^1}$. And for $i > 0$ we have $dF_0(e_i) = d(\Phi_0)d\phi_0^{-1}(e_i) = \frac{\partial}{\partial x^i}$. Thus dF_0 maps basis to basis and so is invertible and yields good coordinates.

Finally, since $\frac{d}{dt}\Phi_t(q)|_{t=0} = X(q)$ and $\frac{\partial}{\partial x^1} = dF_{(x_1,x_2,\ldots,x_n)}(e_1) = \frac{d}{dh}|_{h=0}\Phi_{x_1+h}(\phi^{-1}(0, x^2, \ldots, x^n))$ we see that in the new coordinates $\frac{\partial}{\partial x^1} = X$. QED