AN ALGEBRAIC PERSPECTIVE ON MANIFOLDS, THEIR TANGENT VECTORS, COVECTORS, AND DIFFEOMORPHISMS.

RMONT

The theory of smooth manifolds takes some of its cues from Algebraic Geometry where one replaces a space under study by the ring of functions on that space. Also relevant is the Gelfand-Naimark structure theorem regarding commutative C^* algebras.

1. Ring of Smooth function and its maximal ideal.

Recall that in a ring we can add and multiply. If the ring is commutative then its arithmetic in obeys the rules of addition and multiplication we learned when we first took algebra. If the ring contains a unit 1, and all scalar multiples $c1 = c, c \in \mathbb{R}$ of this unit then the ring forms an *algebra* over $\mathbb{R}$ – a vector space over $\mathbb{R}$ with a multiplication structure on its vectors.

The vector space of all smooth functions on a manifold M forms a commutative ring – an $\mathbb{R}$ algebra – denoted $C^\infty(M)$. Addition and multiplication in this ring is defined as usual, i.e pointwise, so that, for example $(f+g)(p) = f(p)+g(p)$, for $f, g \in C^\infty(M), p \in M$. The unit 1 of the ring is the function identically equal to 1.

We recall that an 'ideal' S of a ring A is a subring $S \subset A$ which is closed under multiplication by elements in A: if $f \in S$ and $g \in A$ then $fg \in S$.

Exer. Let $X \subset M$ be any subset. Show that the set of functions $f \in R$ which vanish on X is an ideal.

Recall that a 'maximal ideal' is an ideal $S \subset A$ with the property that any other ideal $S \subset S' \subset A$ is either equal to S or to all of A.

Basic exercise in ring theory: An ideal $\mathfrak{m} \subset A$ is maximal if and only if the quotient ring $A/\mathfrak{m}$ is a field.

Exercise 1. *For each $p \in M$ the evaluation map $f \mapsto f(p)$ is a ring homomorphism $A \to \mathbb{R}$ whose kernel is m_p the ideal of smooth functions vanishing at p. But then $A/m_p \cong \mathbb{R}$, so by* **Basic Exercise** *, m_p is a maximal ideal.*

Fact/ mini-research project. Every maximal ideal of $C^\infty(M)$ has the form, $\mathfrak{m}_p$, $p \in M$, provided M is compact. If M in non-compact there are other maximal ideals. (See "non-principal ultrafilters".)

Some basic culture. The celebrated Gelfand representation theorem concerns an analogous fact when X is compact Hausdorff and the algebra is the $\mathbb{C}$-algebra $C^0(X, \mathbb{C})$ of continuous $\mathbb{C}$-valued functions. The closed maximal ideals of $C^0(X, , \mathbb{C})$ are of the form $m_p = \{f \in C^0(X, \mathbb{C}) : f(p) = 0\}$. The theorem of Gelfand (or Gelfand-Naimark) asserts that every commutative C^*-algebra A with unit is of the form $C^0(X, \mathbb{C})$ for some compact topological space X. C^* algebras enjoys two additional structures, (i) a normed topology and (2) a *-operation and with these structures $C^0(X, \mathbb{C})$ becomes a "commutative C^* algebra. I will let you look up the definitions. The space X is called the "spectrum" or 'Gelfand dual" of the algebra A.

Example of Gelfand-Naimark. Suppose that $T : \mathbb{C}^n \to \mathbb{C}^n$ is a self-adjoint linear transformation. Let A be the set of all operators $S : \mathbb{C}^n \to \mathbb{C}^n$ which have the form $S = p(T)$ for p a polynomial with complex coefficients. Then A is a commutative C^* algebra, aa subalgebra of the algebra of all linear transformations. The space X is a finite subset of the real line, namely the spectrum (set of eigenvalues) of T.

2. Vector fields as Derivations.

From beginning calculus we know that the operation $f \mapsto Df := df/dx$ is a linear operator on the algebra $A = C^\infty(\mathbb{R})$. In addition to being linear, it satisfies the Liebnitz identity:

$$D(fg) = fDg + gDf, \text{ for all } f, g \in A \tag{1}$$

Definition 1. *Let $\mathbb{K}$ be a field and A a $\mathbb{K}$ algebra with unit. Then a derivation on A is a $\mathbb{K}$-linear operator $D : A \to A$ satisfying the Liebnitz identity , eq (1).*

We can now give a definition of 'vector field on a manifold".

Definition 2. *Let $A = \mathbb{C}^\infty(Q)$ be the algebra of smooth functions on a manifold Q. Then a vector field is a derivation of A. The space of all vector fields forms is denoted by $\chi(Q)$ or by $\Gamma(TQ)$.*

Exercise 2. *Prove that if $D : A \to A$ is a derivation of A, then $D(1) = 0$ where 1 is the unit of the commutative $\mathbb{K}$-algebra A*

Example 1. The partial derivatives $\frac{\partial}{\partial x^i}$ are derivations of $A = \mathbb{C}^\infty(\mathbb{R}^n)$.

Exercise 3. *Prove that If $D : A \to A$ is a derivation of A, and if $h \in A$ then hD is also a derivation, where $(hD)(g) = h(Dg)$. Show that the vector space $Der(A)$ of all derivations of A is a module over A.*

It follows from the previous example and the exercise above that

$$X = \Sigma_{i=1}^n X^i \frac{\partial}{\partial x^i} \tag{2}$$

are derivations of $\mathbb{R}^n$.

Theorem 1. *Any vector field on $\mathbb{R}^n$ can uniquely be expressed in the form of eq (2) where the X^i are smooth functions on $\mathbb{R}^n$*

3. Tangent Space. Cotangent space, algebraic definitions.

If X is a smooth vector field on M, and $q \in M$ then $X(q)$ should be a 'vector' attached at q. The vector space to which it is attached is denoted T_qM and called the tangent space at q. Let us set

$$v = X(q) \in T_qM$$

and agree that the meaning of v is as a linear map $v : \mathbb{C}^\infty(M) \to \mathbb{R}$ defined by $v[f] = X[f](q)$. From $X[fg] = fX[g] + gX[f]$ we see that

$$v[fg] = f(q)v[g] + g(q)v[f] \tag{3}$$

This suggests the definition:

Definition 3. *The tangent space at $q \in M$ is the space of linear functionals $C^\infty(M) \to \mathbb{R}$ satisfying the additional derivation condition of eq (3).*

Now please observe that if $f, g \in \mathfrak{m}_p$, the ideal of functions vanishing at p, then $v[fg] = 0$. The linear span of functions of this form $fg, f, g \in \mathfrak{m}_p$ is denoted $\mathfrak{m}_p^2$ and is a subalgebra: $\mathfrak{m}_p^2 \subset \mathfrak{m}_p \subset C^\infty(M)$.. By linearity v vanishes on $\mathfrak{m}_p^2$. Furthermore, note that if $f \in C^\infty(M)$ then $f - f(q) = f - f(q)1 \in \mathfrak{m}_q$ and that $v[f] = v[f - f(q)]$ since $v[1] = 0$. We have proved:

Proposition 1. *Each $v \in T_qM$ induces a linear map $\mathfrak{m}_p/\mathfrak{m}_q^2 \to \mathbb{R}$, and this linear functional uniquely determines v*

The propositiion asserts that the existence of a canonical linear injection $T_qM \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$. Our next goal is to show that this linear injection is onto, so an isomorphism.

Some terminology is in order.

Definition 4. *The cotangent space at T_q^*M at q is $\mathfrak{m}_q/\mathfrak{m}_q^2$. for the subalgebra of $C^\infty(M)$ consisting of all functions*

Exercise 4. *, For $q = 0 \in M = \mathbb{R}^d$ show that $\mathfrak{m}_o^2$ consists of all smooth functions whose first order Taylor expansion vanishes at 0. More generally, for $k > 0$ an integer show that $\mathfrak{m}_0^{k+1}$ consists of all smooth functions whose kth order Taylor expansion at 0 is identically zero. And show that $C^\infty(\mathbb{R}^d)/\mathfrak{m}_0^{k+1}$ can be identified with the space of degree k polynomials on $\mathbb{R}^d$, i.e., of kth order Taylor approximations.*

As a consequence of the exercise we find:

Proposition 2. *T_q^*M is a real vector space of dimension $n = dim(M)$. Coordinates $x^1, \ldots, x^n$ defined in a nbhd of q induce a basis denoted $dx^1, \ldots dx^n$ and defined within the proof.*

Proof. Let $x^1, \ldots, x^n$ be coordinates centered at p. Extend them to all of M by 'bumping them off" using a bump function β. By this I mean replace x^ii by the function βx^i defined on a nbhd U of p where β is identically one in a nbhd of p and identically 0 outside the coordinate chart, so that we exted βx^i be be zero off this coordinate nbhd. Now, use the same symbol x^i for βx^i. **Basic Taylor series:** If $f(p) = 0$, then near p we have $f = \Sigma a_i x_i + \Sigma b_{ij} x_i x_j + O(x^3)$. Thus $f \equiv \Sigma a_i x^i (mod \mathfrak{m}_p^2)$ and the x^i mod $\mathfrak{m}_p^2$ form a basis for T_p^*M.

The dual basis to dx^i is written $\frac{\partial}{\partial x^i}$. Each $\frac{\partial}{\partial x^i}$ acts on $C^\infty(M)$ by expressing $f \in C^\infty(M)$ in the local coordinates x^i and taking the resulting partial derivative. So if $\phi : \mathbb{R}^n \dashrightarrow M$ is the inverse of the coordinate chart whose components are $(x^1, \ldots, x^n)$ then we set

$$\frac{\partial}{\partial x^i}[f] = \frac{\partial (f \circ \phi)}{\partial x^i}|_{\phi^{-1}(q)}.$$

Now, each $\frac{\partial}{\partial x^i}$ acts as a derivation, and they are linearly independent. We have proved that the canonical map $T_qM \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$ is onto, and hence an isomorphism.

4. Summary

Coordinates x^i at q induce a basis dx^i for the cotangent space T_q^*Q and a dual basis $\frac{\partial}{\partial x^i}$ fo the tangent space T_qQ at q.
****

OVERFLOW ¿...

5. Derivations and the tangent space.

Definition 5. *A derivation v on a commutative algebra A over $\mathbb{R}$ is an $\mathbb{R}$- linear map $v : A \to A$ which in addition satisfies the Leibnitz (product) rule $v(fg) = fv(g) + gv(f)$.*

PROVISIONAL DEF. A vector field on M is a derivation of the commutative ring $C^\infty(M)$.

In $\mathbb{R}^n$. If $X = (X^1, \ldots, X^n) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field then the corresponding derivation is the associated directional derivative, written $f \mapsto X[f]$ where

$$X[f](x) = \Sigma X^i(x) \frac{\partial f}{\partial x^i}|_x := X(x) \cdot \nabla f(x)$$

Localizing at a point p:

Definition 6. *If $\mathfrak{m}_p \subset A$ is a maximal ideal with quotient field $\mathbb{R}$,write $f \mapsto f(p)$ for the canonical identification $A/\mathfrak{m}_p \to \mathbb{R}$. Then by a "derivation at p" we mean a linear functional $v_p : A \to \mathbb{R}$ satisfying $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$.*

To go from vector fields to tangent vectors from this perspective, if $v : C^\infty(M) \to C^\infty(M)$ is a derivation, we define the corresponding tangent vector at p by the formula $v_p(f) = v(f)(p)$.

Exercise. For any derivation, and for any derivation at p we have that $v_p(1) = 1$. HInt: $1 = 1 \cdot 1$. Now use Leibnitz.

Exercise. For any derivation v_p at p we have that $v_p(f) = v_p(f - f(p)1)$ and so v is completely determined by its values on $\mathfrak{m}_p$.

Exercise. For any derivation v_p at p and any $\psi \in \mathfrak{m}_p^2$ we have that $v_p(\psi) = 0$.

Corollary of the last two exercises. Every derivation v_p at p determines, and is completelety determined by, a linear functional $\mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}$. In other words, there is a canonical injective linear map $T_pM \to (\mathfrak{m}_p/\mathfrak{m}_p^2)$.

Proposition 3. *: This map is onto: every linear functional $v \in (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ determines a unique derivation. If x^i are local coordinates near p then the $\frac{\partial}{\partial x^i}$ form a basis for T_pM.*

Methods of proof. Germs, localization ...

6. Germs. Curve germs as tangent vectors.

We would like to do coordinate computations. If x^i are local coordinates at p then dx^i, $i = 1, \ldots, n$ should be a basis for T_p^*M and $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ should be a basis for T_pM. But the x^i are NOT elements of $C^\infty(M)$ since they are only defined in a nbhd of p and not on all of M. We could fix this 'problem' by using bump functions to extend the x^i to all of M but this process obfuscates what is really going on with the cotangent and tangent space. The heart of the matter only depends on things very close to p. Functions do not need to be defined on all of M, only near p, and as far as tangent vectors and covectors at p as they aris from functions or derivations on functions, we do not care at all about the values of f far away from p. To formalize this "not depending on values far from p" we introduce the useful notion of germs.

Let $p \in X$, X a manifold. Let Y be a nother manifold. Write $f : (X, p) \dashrightarrow Y$ to mean that f has domain some nbhd U of p and $f : U \to Y$. If we also fix a point $q \in Y$ then when we write $f : (X,p) \dashrightarrow (Y,q)$ we mean that in addition to $f : (X,p) \dashrightarrow Y$ we have $f(p) = q$.

Definition 7. *Two smooth maps $f, g : (X,p) \dashrightarrow Y$ have the same germ if there is a nbhd V of p on which both f and g are defined and such that such that $f|_V = g|_V$.*

Exercise: The space of all germs of functions $(M,p) \to \mathbb{R}$ forms a local ring, denoted $C^\infty(M)_p$ whose unique maximal ideal consists of the germs of functions vanishing at p which we will continue to write as $\mathfrak{m}_p$.

Definition 8. *Two smooth functions $F, G : (\mathbb{R}^n, 0) \to (\mathbb{R}^d, 0)$ agree to first order if $|F(x) - G(x)| = O(|x|^2)$*

Basic calculus exercise. Two smooth $F, G : (\mathbb{R}^n, 0) \to (\mathbb{R}^d, 0)$ agree to first order iff $DF(0) = DG(0)$ as linear maps $\mathbb{R}^n \to \mathbb{R}^d$.

Definition 9. *Two smooth maps $f, g : (X,p) \dashrightarrow (Y,q)$ "agree to first order" if in some coordinate system $\psi_X : (\mathbb{R}^n, 0) \dashrightarrow (X,p), \phi)Y : (\mathbb{R}^d, 0) \dashrightarrow (Y,q)$ we have that $\phi_Y^{-1} \circ f \circ \phi_X$ and $\phi_Y^{-1} \circ g \circ \phi_X$ agree to first order*

Exercise in def of coordinates. If two smooth maps agree to first order rel. one pair of coordinate systems, then they agree to first order rel. any other compatible pair of coordinates systems.

Exercise. $f \in \mathfrak{m}_p$ agrees to first order with $g \in \mathfrak{m}_p$ if and only if $f - g \in \mathfrak{m}_p^2$.

Consequently, we can define $T_p^* M$ as equivalence classes of germs of functions $f \in \mathfrak{m}_p$ where two functions are equivalent if they agree to first order. By taking $f = x^i$ where $(x^1, \ldots, x^n)$ are coordinates, and setting $dx^i = x^i - x^i(p)$ mod $\mathfrak{m}_p^2$ we get the standard "coordinate basis" for $T_p^* M$.

6.1. **Derivations via Curves. Tangent vectors via curves.** Now that we have the notion "agree to first order" we can define a tentative tangent space $TentativeT_p'' M$ to be the set of equivalence classes curve germs $c : (\mathbb{R}, 0) \dashrightarrow (M, p)$ where we call two such curve germs equivalent if they agree to first order. This definition hides the linear structure. To recove it, we define a map into the derivations at p.

For $c : (\mathbb{R}, 0) \dashrightarrow (M, p)$ a curve germ, show that the operation

$$f \mapsto \frac{d}{dt}|_{t=0} f \circ c$$

defines a derivation at p. Show that the derivation is zero if and only if c is agrees to first order with the constant curve $t \mapsto p$. Show that two curves yield the same derivation if and only they agree to first order.

Notation: $[f - f(p)] mod(\mathfrak{m}_p^2) = df(p)$. $c'(0) = v_p \in T_p M$ is the equivalence class of c and is called the tangent vector to the curve. Then

$$\frac{d}{dt}|_{t=0} f \circ c = df(p) v_p, df(p) \in T_p^* M, v_p = c'(0) \in T_p M$$

As a result we have a well-defined map $TentativeTpM \to Der_p(M)$. This map is injective and onto, consequently allowing us to add curves by adding the corresponding derivations. Finally, by taking the curves to be successively the n coordinate curves $c_i(t) = \psi(te_i) = \psi(0, \ldots, 0, t, 0, \ldots, 0)$, $i = 1, \ldots, n$ we see that $Der_p(M)$ is linearly isomorphic to $(\mathfrak{m}_p / \mathfrak{m}_p^2)^*$, since the n tangent vectors so represented are the partial derivative operators $\frac{\partial}{\partial x^i}$ and these are precisely the dual basis to the coordinate-induced basis dx^i for T_p^*M. This shows that $TentativeTpM$ is canonically isomorphic to our originally defined T_pM.

SUMMARY: We can think of tangent vectors as equivalence classes of curves, or as derivations on functions, or as the dual space to T_p^*M. The space of all tangent vectors at a point to an n-manifold at a point p is an n dimensional real vector space denoted T_pM, with basis $\frac{\partial}{\partial x^i}$, $i = 1, \dots, n$, where x^i are coordinates at p. Composition of a function with a curve realizes the canonical duality pairing $T_p^*M \times T_pM \to \mathbb{R}$.

Definition 10. *Two curve germs $c_1, c_2 : (\mathbb{R}, 0) \dashrightarrow (M, p)$ have the same tangent vector, or "agree to first order" if for all smooth function germs $f : (M, p) \to (\mathbb{R}, 0)$ we have that the function germs $(\mathbb{R}, 0) \dashrightarrow (\mathbb{R}, 0)$ defined by composition, namely, $f \circ c_1$ and $f \circ c_2$ have the same first derivative at 0.*

Exercise 5. *Two curve germs c_1, c_2 agree to first order if and only if their coordinate representatives, $\phi^{-1} \circ c_1, \phi^{-1} \circ c_2 : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ rel. to some coordinate system ϕ centered at p have the same derivative, if and only if for every coordinate rep ψ centered at p we have that $\psi^{-1} \circ c_1, \psi^{-1} \circ c_2 : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ have the same derivative.*

A. Suppose that $f, g \in \mathfrak{m}_p$ are two functions which agree in a neighborhood of p. Then $v_p(f) = v_p(g)$ for any derivation at p.

Proof. $f - g \in \mathfrak{m}_p^2$. (Verify !)

Approach to proof. Localization, using the idea of germs, bump functions centered at p, and coordinate independence.

We will use coordinates and localization to show that the computation at p in an n-manifold M is identical to the computation at 0 in $\mathbb{R}^n$.

We first work in $\mathbb{R}^n$ at 0. Let $v \in \mathbb{R}^n$. Then v defines the curve $c(t) = tv$ thru 0, and, by directional differentiation, a linear functional $v_0 : \mathbb{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ which sends f to $v_0[f] := ddt_{t=0} f(tv)$. From basic calulus v_0 is a derivation at 0. An easy exercise shows that the map $v \mapsto v_0$ is a linear map. In terms of standard coordinates x^i on $\mathbb{R}^n$, if $v = v^i = (v^1, \dots, v^n)$ then $v_0[f] = \Sigma_i = 1^n v^i \frac{\partial f}{\partial x^i}|\{x = 0\}$ showing that this map is an isomorphism from $\mathbb{R}^n$ to $T_0(\mathbb{R}^n) := (\mathfrak{m}_0/\mathfrak{m}_0^2)^*$.

Now, if $p \in M = M^n$, we can choose a small coordinate nbhd U of p and a bump function ϕ which is identically 1 near p and identically 0 off of U. Then $v(\phi f) = \phi(p)v(f) + f(p)v(\phi)$. But near p we have $\phi\phi = \phi = 1$ so that $v(\phi) = 2\phi(p)v(\phi) = 2v(\phi) = 0$. Thus $v(\phi f) = v(f)$.

SUMMARY: Cutting off f by multiplying by a bump function with support near p does not change the value of $v_p(f)$.

Coordinate, or curve exercises.

...

7. Automorphisms

An automorphism of a commutative algebra over $\mathbb{R}$ is an algebra homomomorphism $A \to A$ which is invertible. The set of all automorphisms of A forms a group $Aut(A)$.

In the case $A = C^\infty(M)$ we have a natural anti-homomorphism $Diff(M) \to Aut(A)$: which sends a diffeo $\phi : M \to M$ to the automorphism of "pullback by ϕ", which is to say, the operation on smooth functions f given by $f \mapsto \phi^* f = f \circ \phi$.

A one-parameter subgroup of automorphisms is a homomorphism from the real line into $Aut(A)$. Thus it is a one-parameter family of maps $\psi_t : A \to A$ such that $\psi_0 = id$ and $\psi_t \circ \psi_s = \psi_{t+s}$. Assuming that ψ_t is differentiable, we compute that its derivative at $t = 0$ is a derivation.

In the case of $A = C^\infty(M)$, we have that $\phi_t = exp(tv)-$ the flow of a vector field v - is a one-parameter subgroup of diffeos, and hence of automorphisms.

Theorem? [Ref?] For M compact, every automorphism of $C^\infty(M)$ arises as pullback by a diffeo. Every derivation of A arises from a vector field.

8. Modules and Bundles

Recall the notion of a module over a ring. A module M is to a ring A as a vector space V is to its underlying field of scalars F. What this mean is that M forms an additive group, and that A acts on M by a "scalar multiplication": $A \times M \to M$, written $(a, m) \mapsto (am)$ satisfying the obvious axioms: $a(m_1 + m_2) = am_1 + am_2, a_1(a_2 m) = (a_1 a_2)m$.

The smooth sections $\Gamma(E)$of a smooth vector bundle $E \to M$ form a module over $A = C^\infty(M)$. We would like to say that "modules over $C^\infty(M)$ are in natural bijection with smooth vector bundles over M"'. This is not true. But it becomes true once we require our modules to be "projective" and "finitely generated" . A version of the Serre-Swan theorem asserts that over the category of smooth compact manifolds, this bijection holds. In other words, if $Vect(M)$ is the category of smooth finite-dimensional vector bundles over a compact manifold space M, then the map $E \mapsto \Gamma(E)$ which sends a vector bundle to its space of smooth sections, defines a functor *onto* the space of finitely generated projective modules over $C^\infty(M)$. I think you can guess what it means for a module to be "finitely generated". (It is essentially the same meaning as that of a a vector space being finite dimensional.) The definition of "projective" is more subtle and I will let you look up the various equivalent definitions. Also, please see the wiki entry on "Serre-Swan theorem" , which is quite decent.

9. Tangent and Cotangent Functor

If $F : M \to N$ is smooth, we have an associated smooth map $TF : TM \to TN$ which is a vector bundle map over f. It sends $T_m M$ to $T_{F(m)}N$ linearly. This linear map based at m is written variously dF_m or F_{*m}, as in

$$dF_m : T_m M \to T_{F(m)}N.$$

On the level of curves it can be defined as the map sending $v_m = c'(0) \in T_m M$ to the derivative of the curve $F \circ c$ at $t = 0$. Note that this image curve $F \circ c$ indeed passes through $F(m)$ at $t = 0$.

The chain rule becomes the obvious thing: if $F : M \to N$ and $G : N \to Y$ then $T(G \circ F) = TG \circ TF : TM \to TY$, and is a vector bundle map over $G \circ F$.

EXER. Define the corresponding map on the cotangent fibers: $dF^*_m : T^*_{f(m)}N \to T^*_m M$. Verify that dF^*_m is the linear algebraic dual of dF_m.