PROBLEM A BACKGROUND. If f is a smooth function on an n-dimensional manifold M then its differential df is well-defined, and given by the coordinate expression

$$df = \Sigma_{i=1}^{n} \frac{\partial f}{\partial x^i}|_p dx^i.$$

For $p \in M$ we have $df(p) \in T_p^* M$. s

The goal of this exercise is to show that the corresponding "Hessian of f", whose coordinate expression at a point $p \in M$ *should be*

$$\Sigma_{i,j} \frac{\partial^2 f}{\partial x^i x^j}|_p dx^i dx^j$$

is only well-defined at critical points of f. (Here the "$|_p$" means evaluate the partial derivatives at the coordinate point corresponding to p.) This expression is to be viewed as a quadratic form (or bilinear symmetric form) on $T_p M$,

PROBLEM A) Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be two coordinate systems centered at p. ('centered' here means $x^i(p) = 0 = y^i(p)$.) Suppose that $df(p) = 0$. Show that $\Sigma_{i,j} \frac{\partial^2 f}{\partial x^i x^j}|_p dx^i dx^j = \Sigma_{i,j} \frac{\partial^2 f}{\partial y^i y^j}|_p dy^i dy^j$

PROBLEM B BACKGROUND. The normal form theorem for functions (proved in class) asserts that if $df(p) \neq 0$ then there are coordinates $(x^1, \ldots, x^n)$ centered at p such that when expressed in these coordinates we have that $f = x_1 + C$ where C is the constant $f(p)$.

PROBLEM B (i) Suppose that $df(p) \neq 0$. What is the Hessian of f with respect to these normal form coordinates ?

PROBLEM B (ii) Suppose that $df(p) \neq 0$. Let M_{ij} be the components of an arbitrary n by n real symmetric matrix. Show that there are coordinates $y^1, y^2, \ldots, y^n$ centered at p so that the Hessian of f with respect to the y^i at p is $\Sigma M_{ij} dy^i dy^j$. HINT: Look for a change of coordinates from the normal form coordinates x^i to new coordinates y^i which has the form $y^i = x^i + Q^i(x^1, \ldots, x^n)$ with the Q^i quadratic functions in the x^j.

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FOOD FOR THOUGHT. An n by n real symmetric matrix defines a real quadratic form on $\mathbb{R}^n$ by $Q(x^1, \ldots, x^n) = \Sigma_{i,j} M_{ij} x^i x^j$ Not all quadratic forms Q on an n dimensional real vector space are equivalent under change of basis. There are two invariants of a real quadratic form Q: its rank and its index, both being integers between 0 and n. Look up the definitions of these invariants if you do not know them. What is the rank and index of the zero form corresponding to $M_{ij} = 0$? Can you list all quadratic forms, up to change of basis, on $\mathbb{R}^2$?