HW 2.

2.1[Hyperelliptic Riemann surface]

Consider the solution set to the equation $w^2 = P(z)$ in $\mathbb{C}^2$. Here (z, w) are standard complex coordinates on $\mathbb{C}^2$ and $P(z)$ is a polynomial in the complex variable z. By following the steps below, prove that this solution set is a smooth surface provided that the zeros of P are distinct.

2.1A. In order to obtain the proof, first formulate or even prove an implicit function theorem for functions of several complex variables.

2.1B. Note that if (z_0, w_0) lies on the surface then $w_0 = 0$ iff $P(z_0) = 0$.

Now use your IFT from 1A, dividing into cases according to whether $P(z_0) = 0$ or not.

2.1C. Away from zeros of P. If (z_0, w_0) lies on the surface and $P(z_0) \neq 0$ prove that there exist two smooth function $w_\pm = w(z)$ defined near z_0, related to each other by $w_-(z) = -w_+(x)$, and such that in a neighborhood of z_0 we have that (z_0, w) lies on the surface if and only if $w(z)$ is one of $w_\pm(z)$.

2.1D. $P(z_0) = 0$. You can no longer parameterize as $w = w(z)$. Instead show that you can parameterize the surface by w by looking for smooth $z = z(w)$ defined near $(z, w) = (z_0, 0)$.

2.1C. Consider the map F which is the restriction of the projection $(z, w) \mapsto z$ to the hyperelliptic surface $z^2 = P(z)$. Is F smooth? If 'yes' find its critical points. If 'no' describe the points where it fails to be smooth.

2.2. Consider the map $\Phi : \mathbb{RP}^{n-1} \to Sym(n)$ which sends a line $[v] \in \mathbb{RP}^{n-1}$ to the orthogonal projection $\Phi([v])$ onto that line.

2.2 a) Show that Φ is one-to-one.

2.2 b) Express Φ in homogeneous coordinates $\Phi([x_1, \ldots, x_n])_{ij} =?$ – a symmetric n by n matrix depending rationally on the homogeneous coordinates x_i.

2.2 c) Show that Φ is smooth.