HW Assignment 1.

1.1 A. Prove that the locus $xy = 0$ in the plane, is NOT a topological manifold.

B. For what values of c is the locus $xy = c$ a smooth manifold?

1.2. Set-up: Consider the space of all lines in the plane. Since a line can be given in slope-intercept form: $y = mx + b$, we can think of the slope m and intercept b as the coordinates of the line. Thus the space of lines must form a two-dimensional manifold. But the slope-intercept form misses one family of lines, the vertical lines, which have infinite slope. We can write these lines in the form $x = B$, B a constant, and as such they form part of the family of lines expressed in the form $x = My + B$.

1.2 A. Compute the overlap map $(m, b) \mapsto (M, B)$.

1.2B. Prove that the space of all lines in the plane , with the manifold structure you just deduced, is diffeomorphic to the tangent bundle of the circle, and that this tangent bundle is in turn diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

1.3. A holomorphic function $F : \mathbb{C} \to \mathbb{C}$ can be viewed as a function from the plane $\mathbb{R}^2$ to itself, by having $z = x + iy \in \mathbb{C}$ correpsond to $(x, y) \in \mathbb{R}^2$. What is the relation between the complex derivative $dF/dz(z) \in \mathbb{C}$ and the real linearization, which is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$?

1.4. Let $gl(n)$ denote the vector space of n by n real matrices, and $Sym(n)$ the vector space of symmetrid n by n real matrices.

a) What is the dimension of $gl(n)$?

b) What is the dimension of $Sym(n)$?

c) Compute the derivative of the map $F(A) = AA^t$ at the value $A = Id$.

d) Think of this F as a map $F : gl(n) \to Sym(n)$ Show that $Id \in Sym(n)$ is a regular VALUE of F.

e) Conclude that $F^{-1}(Id)$ is a smooth manifold and compute its dimension. (Terminology: This level set $F^{-1}(Id)$ is known as the orthogonal group and denoted by $O(n)$. It is the group of linear isometries $\mathbb{R}^n \to \mathbb{R}^n$.)