FINAL. DUE FRIDAY DEC 9 BY NOON.

Instructions. I urge you to TeX up your solutions. This problem concerns the Hopf fibration.

BACKGROUND and NOTATION.

Let x_1, x_2, x_3, x_4 be standard coordinates on $\mathbb{R}^4$ so that S^3 is defined by $\{(x_1, x_2, x_3, x_4) : \Sigma_{i=1}^4 x_i^2 = 1\}$ and the Euclidean metric is $ds_E^2 = \Sigma_{i=1}^4 dx_i^2$. Stereographic projection is the map $F : S^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ given by

$$F(x_1, x_2, x_3, x_4) = (u_1, u_2, u_3); u_i = \frac{x_i}{1 - x_4}, i = 1, 2, 3,$$

The standard ("round") metric ds^2 on S^3 is the restriction of ds_E^2 to vectors tangent to the sphere.

A) i) Show that $ds^2 = f(u_1, u_2, u_3)(du_1^2 + du_2^2 + du_3^2)$ and compute the function f (the "conformal factor").

ii) Explain why (i) proves that stereographic projection preserves angles: the angle at $x \in S^3$ between vectors $v, w \in T_x S^3$ equals the angle in $\mathbb{R}^3$ between $dF_x v, dF_x w$.

B) BACKGROUND and NOTATION. Form $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$ and define

$$\pi : S^3 \to \mathbb{R}^3$$

by

$$\pi(x_1, x_2, x_3, x_4) = (w_1, w_2, w_3); w_1 + iw_2 = 2z_1\bar{z}_2, w_3 = |z_1|^2 - |z_2|^2$$

(The map π is called the "Hopf fibration".)

i) Verify the image of π is the two-sphere $S^2 \subset \mathbb{R}^3$.

ii) Verify π is a submersion when viewed as a map $S^3 \to S^2$.

iii) S^1 acts on S^3 by $e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$. Verify that for each $w \in S^2$ the fiber $\pi^{-1}(w)$ is a single orbit of this circle action.

iv). Prove that the quotient space S^3/S^1 is diffeomorphic to S^2.

v) Let V be a smooth vector field on S^3. Prove that if if $[\frac{\partial}{\partial\theta}, V] = 0$ then $\pi_* V$ makes sense as a vector field on S^2 (some authors say "V is π-projectible") Here $\frac{\partial}{\partial\theta}$ is the infinitesimal generator of the Hopf flow of (iii).

C): BACKGROUND: A "geometric two-sphere" is an embedded two-sphere in S^3 obtained by intersection S^3 by an affine plane $Ax_1 + Bx_2 + Cx_3 + Dx_4 = E$.

i) Prove that if Σ is a geometric two-sphere in S^3 then its image $F(\Sigma)$ is either a sphere or a plane in $\mathbb{R}^3$.

ii) Define a 'geometric circle' in S^3 . Verify that the image of a geometric circle in S^3 is either a circle or a line in $\mathbb{R}^3$.

iii) Sketch, in $\mathbb{R}^3$, the image of a few Hopf fibers, i.e. sketch the loci $F(\pi^{-1}(w))$ for a few different choices of w.

D) BACKGROUND: A "Clifford torus" is a surface in S^3 defined by $|z_1|^2 = a, |z_2|^2 = b$ where a, b are positive constants with $a + b = 1$.

i) Sketch the image under F of a Clifford torus.

ii) The circle action leaves the Clifford torii invariant, hence these torii are fibered by Hopf fibers. Sketch this fibration by circles of your torus from (i).