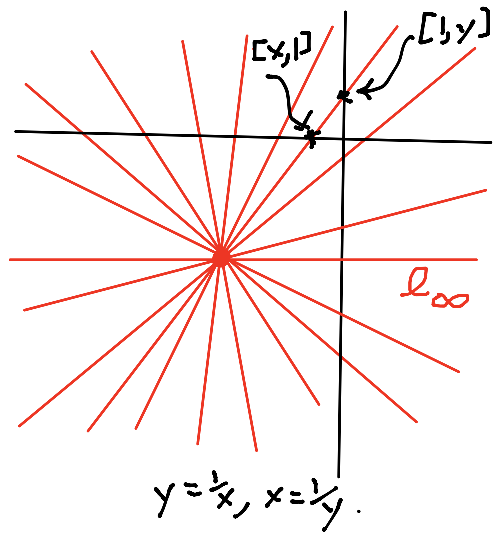
$\mathbb{RP}^1$ is the space of lines through the origin in $\mathbb{R}^2$. To coordinatize it, it is useful to view it as the quotient space of $\mathbb{R}^2 \setminus \{(0,0)\}$ by the equivalence relation $(x,y) \sim (u,v) \iff span(x,y) = span(u,v)$. We write the quotient map

$$\mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{RP}^1; \quad (X,Y) \mapsto [X,Y].$$

Observe that $[X,Y] = [\lambda X, \lambda Y]$ for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$. We call $[X,Y]$ the HOMOGENEOUS COORDINATES of the line ℓ.

AFFINE CHARTS.



$\mathbb{RP}^1$ is the union of two large open sets, called "affine spaces" , U_x and U_y:

$$\mathbb{RP}^1 = U_x \cup U_y.$$

We give three equivalent descriptions of U_x.

- U_x consists of all lines through the origin which intersect the line $Y = 1$.
- U_x consists of all lines through the origin which can be expressed by an equation of the form

$$X = MY, M \in \mathbb{R}.$$

- U_x consists of all lines EXCEPT the x-axis $Y = 0$.

The intersection point of a line $\ell \in U_x$ with the line $Y = 1$ is $(M, 1) = (x, 1)$. We call x the "affine coordinate" of ℓ in the chart U_x. The corresponding coordinate chart is then

$$\phi_x : U_x \to \mathbb{R}; [X, Y] \mapsto x = X/Y = M.$$

with inverse map

$$\phi_x^{-1} : \mathbb{R} \to U_x; \phi_x^{-1}(x) = [x, 1].$$

As $x \to \infty$ the line $[x, 1]$ tends to the line $\ell_\infty = [1, 0]$, which is the X-axis. Thus, we can say $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$.

Chart covering infinity To cover the rest of $\mathbb{RP}^1$, which is to say to cover the missing ℓ_∞ , we switch the roles of X and Y. Thus let

$$U_y = \{[X, Y]; X \neq 0\} = \mathbb{RP}^1 \setminus \{\ell_0\} = \text{ lines intersecting the line} X = 1.$$

Here ℓ_0 is the Y-axis, which is the locus $X = 0$. Any line $\ell \in U_y$ can be expressed in standard slope form as

$$Y = mX$$

and, since $Y \neq 0$ somewhere on ℓ we have

$$\ell = [X, Y] = [1, Y/X] = [1, y]$$

allowing us to define the affine coordinate y according to

$$\phi_y : U_y \to \mathbb{R}; [X, Y] \mapsto y = Y/X.$$

with inverse map

$$\phi_y^{-1} : \mathbb{R} \to U_x; \phi_y^{-1}(y) = [1, y].$$

Note : the slope m is this affine coordinate:

$$m = Y/X = y$$

Coordinate overlap or Transition function. $U_x \cap U_y =$ lines through $(0, 0)$ except the Y-axis and the X-axis. Alternatively:

$$U_x \cap U_y = \mathbb{RP}^1 \setminus \{\ell_\infty, \ell_0\}$$

The coordinate overlap map $\phi_y \circ \phi_x^{-1}$ has domain $\mathbb{R} \setminus \{0\}$ and is given by

$$x \mapsto [x, 1] = [1, 1/x] = [1, y] \mapsto y = 1/x.$$

SEE FIGURE. Alternatively, any line $\ell \in U_x \cup U_y$ is expressed in homogeneious coordinates as an $[X, Y]$ with $X \neq 0$ and $Y \neq 0$ so we can write ℓ as $[X/Y, 1]$ and also as $[1, Y/X]$ Setting $x = X/Y, y = Y/X$ we see that the overlap is $x \mapsto 1/x$.

OTHER FIELDS. All these considerations work with any field $\mathbb{K}$ replacing $\mathbb{R}$. For our purposes, $\mathbb{C}$ is a good field. We have $\mathbb{CP}^1 =$ complex lines through the origin in $\mathbb{C}^2$. Also $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} = S^2 =$ Riemann sphere.