MATH 208 - HW # 7

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Problem 1. Let $\alpha = -y\mathrm{d}x + \mathrm{d}z$, a one-form on $\mathbb{R}^3$. Find non-vanishing vector fields V on $\mathbb{R}^3$ s.t. $\mathcal{L}_V\alpha = 0$. Find other vector fields s.t. $\mathcal{L}_V\alpha = f\alpha$, $f \neq 0$ a function.

Solution 1. We take $V = V^1\partial_x + V^2\partial_y + V^3\partial_z$ to be an arbitrary vector field. The Lie derivative can be computed via:

$$\mathcal{L}_V\alpha = \left(V^j\frac{\partial\alpha_i}{\partial x^j} + \alpha_j\frac{\partial V^j}{\partial x^i}\right)\mathrm{d}x^i$$

With $\alpha_2 = 0$ and the derivative of α_3 being zero throws out a lot of terms providing:

$$\mathcal{L}_V\alpha = \Big(-y\partial_x V^1 - V^2 + \partial_x V^3\Big)\mathrm{d}x + \Big(-y\partial_y V^1 + \partial_y V^3\Big)\mathrm{d}y + \Big(-y\partial_z V^1 + \partial_z V^3\Big)\mathrm{d}z$$

- For $\mathcal{L}_V\alpha = 0$ we want the vector field to satisfy:

$$\partial_x\Big(-yV^1 + V^3\Big) = V^2$$
$$\partial_y\Big(-yV^1 + V^3\Big) = -V^1$$
$$\partial_z\Big(-yV^1 + V^3\Big) = 0$$

- For $\mathcal{L}_V\alpha = f\alpha$ we want the vector field to satisfy:

$$\partial_x\Big(-yV^1 + V^3\Big) = V^2 - yf$$
$$\partial_y\Big(-yV^1 + V^3\Big) = -V^1$$
$$\partial_z\Big(-yV^1 + V^3\Big) = f$$

Problem 2. Repeat the previous problem with the one-form $\alpha = \mathrm{d}z$.

Solution 2. We take $V = V^1\partial_x + V^2\partial_y + V^3\partial_z$ to be an arbitrary vector field. The Lie derivative can be computed via:

$$\mathcal{L}_V\alpha = \left(V^j\frac{\partial\alpha_i}{\partial x^j} + \alpha_j\frac{\partial V^j}{\partial x^i}\right)\mathrm{d}x^i$$

With $\alpha_1 = \alpha_2 = 0$ and the derivative of α_3 being zero throws out a lot of terms providing:

$$\mathcal{L}_V\alpha = \partial_x V^3\mathrm{d}x + \partial_y V^3\mathrm{d}y + \partial_z V^3\mathrm{d}z$$

- For $\mathcal{L}_V\alpha = 0$ we want the vector field to satisfy $\partial_x V^3 = \partial_y V^3 = \partial_z V^3 = 0$. It easily follows that $V^3 = c$ for some $c \in \mathbb{R}\setminus\{0\}$. Therefore, any vector field of the form $V = V^1\partial_x + V^2\partial_y + c\partial_z$ ensures $\mathcal{L}_V\alpha = 0$.
- For $\mathcal{L}_V\alpha = f\alpha$ we want the vector field to satisfy $\partial_x V^3 = \partial_y V^3 = 0$ and $\partial_z V^3 = f$. It easily follows that $V^3 = g(z)$ where $g'(z) = f$. Therefore, any vector field of the form $V = V^1\partial_x + V^2\partial_y + g(z)\partial_z$ ensures $\mathcal{L}_V\alpha = f\alpha$.