MATH 208 - HW # 6

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October 23, 2017

Problem 1. Look at all lines in $\mathbb{R}^2$ and note that it forms a manifold. What is its dimension? Provide explicit charts.

Solution 1.

- We note that the dimension of $\mathbb{RP}^1$ came out to be one since it only required information about the slope to identify a unique line in the space. Building off of this we can say that our manifold, M, is characterized as $M \cong \mathbb{RP}^1 \times \mathbb{R}$ because it needs the information about the slope from $\mathbb{RP}^1$ and another value for the intercept of the line. Therefore the dimension of M must be two.
- For $\mathbb{RP}^1$ it took two charts to cover the space since one line would always be missed by any chart. Using this as motivation we can take the two lines $x = 0$ and $y = 0$ and notice that it takes two charts to cover the space.
  - To obtain all non-vertical lines we define the chart $f : U \subseteq M \to \mathbb{R}^2$ given by:
$$f : (m, c) \to \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\}$$
  - To obtain all non-horizontal lines we define the chart $g : V \subseteq M \to \mathbb{R}^2$ given by:
$$g : (m, c) \to \{(x, y) \in \mathbb{R}^2 \mid x = my + b\}$$

  Intuition tells us that the overlap map will consist of all lines having non-zero/non-infinite slope. To see this explicitly we have:
$$g \circ f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{given by} \quad \{(x, y) \in \mathbb{R}^2 \mid y = mx + b\} \to \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x = \frac{1}{m}y - \frac{b}{m} \text{ and } m \neq 0\right\}$$
- It is also interesting to note that $M \cong S^1 \times \mathbb{R}$. The reason behind this is that $\mathbb{RP}^1 \cong S^1$. To show this explicitly recall the stereographic atlas $\{(U_N, \phi_N), (U_S, \phi_S)\}$ for $U_N = S^1 \setminus \{(0, 1)\}$ and $U_S = S^1 \setminus \{(0, -1)\}$ where the maps are given by:
$$\phi_N : (x, y) \to \frac{x}{1 - y} \quad \text{and} \quad \phi_S : (x, y) \to \frac{x}{1 + y}$$
  For $\mathbb{RP}^1$ we use the atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ where $U_1 = U_2 = \mathbb{R} \setminus \{0\}$. The maps are given by:
$$\phi_1 : \{(x, y) \in \mathbb{R}^2 \mid y = mx\} \to \frac{y}{x} \quad \text{and} \quad \phi_2 : \{(x, y) \in \mathbb{R}^2 \mid x = my\} \to \frac{x}{y}$$
  Now we define the mapping:
$$\psi : \mathbb{RP}^1 \to S^1 \quad \text{given by} \quad \psi(l) = \begin{cases} (\phi_N^{-1} \circ \phi_1)(l) & \text{if} \quad l \in U_1 \\ (\phi_S^{-1} \circ \phi_2)(l) & \text{if} \quad l \in U_2 \end{cases}$$
  which is a diffeomorphism between $\mathbb{RP}^1$ and S^1.

Problem 2. Prove that multiplication $(A, B) \to AB$ is a smooth map $SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \to SO(3, \mathbb{R})$.

Solution 2.

- First we notice that $SO(3, \mathbb{R}) \times SO(3, \mathbb{R}) \hookrightarrow gl(3, \mathbb{R}) \times gl(3, \mathbb{R})$ via the identity mapping. Furthermore, the identity mapping is continuous and injective telling us that $SO(3, \mathbb{R}) \times SO(3, \mathbb{R})$ is an embedded manifold in $gl(3, \mathbb{R}) \times gl(3, \mathbb{R})$.
- Take $A, B \in SO(3, \mathbb{R})$ and notice $\det(AB) = \det(A)\det(B) = 1$ and $(AB)^{-1} = B^{-1}A^{-1} = B^TA^T = (AB)^T$. This tells us that $\mathfrak{Im}(f) \subseteq SO(3, \mathbb{R})$ where f represents the matrix multiplication. To show inclusion in the other direction we consider the surjectivity of the map. For any $R \in SO(3, \mathbb{R})$ we can always find the pair $(R, I) \in SO(3, \mathbb{R}) \times SO(3, \mathbb{R})$ s.t. $RI = R$ where I is the identity matrix. Thus, $\mathfrak{Im}(f) = SO(3, \mathbb{R})$.

- Now we show that $f : gl(3,\mathbb{R}) \times gl(3,\mathbb{R}) \to gl(3,\mathbb{R})$ is a smooth map from which it will follow that it is also smooth on the multiplication of rotation matrices, an embedded manifold. Explicitly we have:

$$AB = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \begin{pmatrix} b_1 & b_4 & b_7 \\ b_2 & b_5 & b_8 \\ b_3 & b_6 & b_9 \end{pmatrix} = \begin{pmatrix} \leftarrow & \overrightarrow{v_1} & \rightarrow \\ \leftarrow & \overrightarrow{v_2} & \rightarrow \\ \leftarrow & \overrightarrow{v_3} & \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \overrightarrow{w_1} & \overrightarrow{w_2} & \overrightarrow{w_3} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \langle v_1, w_1\rangle & \langle v_1, w_2\rangle & \langle v_1, w_3\rangle \\ \langle v_2, w_1\rangle & \langle v_2, w_2\rangle & \langle v_2, w_3\rangle \\ \langle v_3, w_1\rangle & \langle v_3, w_2\rangle & \langle v_3, w_3\rangle \end{pmatrix}$$

So using the charts (W_1, ψ_1) and (W_2, ψ_2) where:

$$\psi_1 : W_1 \to \mathbb{R}^{18} \quad \text{given by} \quad \psi_1(A, B) = (a_1, \ldots, a_9, b_1, \ldots, b_9)$$
$$\psi_2 : W_2 \to \mathbb{R}^{9} \quad \text{given by} \quad \psi_2(C) = (c_1, \ldots, c_9)$$

gives us a map from $\zeta = \psi_2 \circ f \circ \psi_1^{-1} : \mathbb{R}^{18} \to \mathbb{R}^9$:

$$\zeta(a_1, \ldots, a_9, b_1, \ldots, b_9) = \begin{pmatrix} \langle v_1, w_1\rangle \\ \langle v_1, w_2\rangle \\ \langle v_1, w_3\rangle \\ \langle v_2, w_1\rangle \\ \langle v_2, w_2\rangle \\ \langle v_2, w_3\rangle \\ \langle v_3, w_1\rangle \\ \langle v_3, w_2\rangle \\ \langle v_3, w_3\rangle \end{pmatrix}$$

Since each component of ζ is smooth, then so is ζ.

Problem 3. Prove that inversion $A \to A^{-1}$ is a smooth map $SO(3,\mathbb{R}) \to SO(3,\mathbb{R})$.

Solution 3.

- First we notice that $SO(3,\mathbb{R}) \hookrightarrow GL(3,\mathbb{R})$ via the identity mapping. Furthermore, the identity mapping is continuous and injective telling us that $SO(3,\mathbb{R})$ is an embedded manifold in $GL(3,\mathbb{R})$.
- Take $A \in SO(3,\mathbb{R})$ and notice $\det(A^{-1}) = \frac{1}{\det(A)} = 1$ and $(A^{-1})^T = (A^T)^{-1} = (A^{-1})^{-1}$. This tells us that $\mathfrak{Im}(g) \subseteq SO(3,\mathbb{R})$ where g represents the matrix inversion. To show inclusion in the other direction we consider the surjectivity of the map. For any $R \in SO(3,\mathbb{R})$ we can always find a unique inverse $R^{-1} \in GL(3,\mathbb{R})$ s.t. $RR^{-1} = R^{-1}R = I$. Furthermore, if R is rotation, then so is the inverse. Thus, $\mathfrak{Im}(f) = SO(3,\mathbb{R})$.
- Now we show that $g : GL(3,\mathbb{R}) \to GL(3,\mathbb{R})$ is a smooth map from which it will follow that it is also smooth on rotation matrices, an embedded manifold. Explicitly we have:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{pmatrix}$$

where M_{ij} represents the determinant of the 2×2 matrix that results from deleting row i and column j of A. This tells us that matrix inversion can be thought of as a map from $\mathbb{R}^9$ to $\mathbb{R}^9$ that sends $(a_1, \ldots, a_9)$ to rational functions of those inputs. Since the determinant is non-zero, we can say that each rational function is smooth implying that g is a smooth map.

Problem 4. Let:

$$E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Prove that $X(g) = gE_3$ defines a smooth vector field on $SO(3,\mathbb{R})$.

Solution 4. For an arbitrary $g \in SO(3,\mathbb{R})$ we can see the action of the derivation directly:

$$X(g) = gE_3 = \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g_2 & -g_1 & 0 \\ g_5 & -g_4 & 0 \\ g_8 & -g_7 & 0 \end{pmatrix}$$

Therefore, the corresponding vector field has to take the form $V(x,y) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. The correspondence comes from observing the behavior of the column vectors. Furthermore, since each component function is smooth, so is the vector field.

Problem 5. Prove that the map $\pi : SO(3,\mathbb{R}) \to S^2$ given by $\pi(g) = ge_3$ is a submersion.

Solution 5. Notice what this map provides explicitly:

$$\pi(g) = \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g_3 \\ g_6 \\ g_9 \end{pmatrix}$$

Using the fact that the column vectors of any rotation matrix make up an orthonormal set implies $g_3^2 + g_6^2 + g_9^2 = 1$. Thus, $\pi(g) \in S^2$. To show that this map is a submersion means we have to show the Jacobian is surjective. To accomplish such a task we need to consider the lifting:

$$\begin{array}{lcl} GL(3,\mathbb{R}) & \xrightarrow{\tilde{\pi}} & (\mathbb{R}^3)^\times \\ \kappa\uparrow & & \uparrow\alpha \\ SO(3,\mathbb{R}) & \xrightarrow{\pi} & S^2 \end{array}$$

where $\kappa(R) = R$, $\tilde{\pi}(A) = Ae_3$, and $\alpha(\overrightarrow{p}) = \overrightarrow{p}$. Since $SO(3,\mathbb{R}) \hookrightarrow GL(3,\mathbb{R})$ and $S^2 \hookrightarrow (\mathbb{R}^3)^\times$ via the identity maps, they are embedded manifolds. Furthermore, we can see that:

$$J(\tilde{\pi}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which has full rank showing that $\tilde{\pi}$ is a submersion of $GL(3,\mathbb{R})$ into $(\mathbb{R}^3)^\times$. Therefore, π is also a submersion since it inherits the behavior from $\tilde{\pi}$.

Problem 6. As one of the HWs the coordinate transition map $x \to \frac{1}{x}$ came up for switching from one chart to another in the standard affine charts of $\mathbb{RP}^1$. Instead of viewing this map passively as a change of coordinates, we can look at it actively, as a transformation of $F : \mathbb{RP}^1 \to \mathbb{RP}^1$, defined within the single affine chart $Y \neq 0$. Find the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ which induces the map $f(x) = \frac{1}{x}$ in this standard affine chart.

Solution 6.

- From the viewpoint of $X = 1$ we have to satisfy the mappings $(1,m) \to \left(\frac{1}{m}, 1\right)$ and $(1,0) \to (0,1)$. The linear transformation must therefore take the form:

  $$L = \begin{pmatrix} 0 & \frac{1}{m^2} \\ 1 & 0 \end{pmatrix}$$

  which misses the infinite point $(X = 0)$.
- From the viewpoint of $Y = 1$ we have to satisfy the mappings $\left(\frac{1}{m}, 1\right) \to (1,m)$ and $(0,1) \to (1,0)$. The linear transformation must therefore take the form:

  $$L = \begin{pmatrix} 0 & 1 \\ m^2 & 0 \end{pmatrix}$$

  which misses the infinite point $(Y = 0)$.