MATH 208 - HW # 6 - Corrections

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Problem 1. Let:

$$E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Prove that $X(g) = gE_3$ defines a smooth vector field on $SO(3, \mathbb{R})$.

Solution 1. Begin by considering the map $f : GL(3, \mathbb{R}) \to SO(3, \mathbb{R})$ given by $f(A) = AA^T$. We can explicitly compute the derivative as:

$$\begin{aligned} f'_A(h) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} f(A + \epsilon h) \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} (A + \epsilon h)(A + \epsilon h)^T \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} (AA^T + \epsilon(Ah^T + hA^T) + \epsilon^2 hh^T) \\ &= \Big((Ah^T + hA^T) + 2\epsilon hh^T\Big)\Big|_{\epsilon=0} \\ &= Ah^T + hA^T \end{aligned}$$

If we restrict $A \in SO(3, \mathbb{R})$, then $f(A) = I$ and consequently $Ah^T + hA^T = 0$. Furthermore, if $A = I$, then we actually end up with $h^T = -h$ implying $h \in \mathfrak{so}(3, \mathbb{R})$. Thus, 3×3 skew-symmetric matrices span the tangent space for $SO(3, \mathbb{R})$ (naturally dimensions also match up). Knowing this information tells us that a vector field will consist of component functions tagged to a direction given by a skew-symmetric matrix. In fact, $E_3 \in \mathfrak{so}(3, \mathbb{R})$ which tells us $X(g)$ is a vector field. To see that is smooth, we just observe that the action of the coordinate function is matrix multiplication which has already been proven to be smooth.

Problem 2. Prove that the map $\pi : SO(3, \mathbb{R}) \to S^2$ given by $\pi(g) = ge_3$ is a submersion.

Solution 2. Notice what this map provides explicitly:

$$\pi(g) = \begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g_3 \\ g_6 \\ g_9 \end{pmatrix}$$

Using the fact that the column vectors of any rotation matrix make up an orthonormal set implies $g_3^2 + g_6^2 + g_9^2 = 1$. Thus, $\pi(g) \in S^2$. To show that this map is a submersion means we have to show the Jacobian is surjective. First we prove the Equivariant Rank Theorem: Let M and N be smooth manifolds and G a Lie group. Suppose $F : M \to N$ is a smooth map that is equivariant with respect to a transitive smooth G-action on N and M. Then F has constant rank.

- Pick an arbitrary point $p \in M$ and let ϕ and ψ denote the G-actions on M and N respectively. Due to the transitivity of G we can always find a $g \in G$ s.t. $\phi_g(p) = q$ and $\psi_g(F(p)) = F(q)$, which combined with $\psi_g \circ F = F \circ \phi_g$ provides the commutative diagram:

$$\begin{array}{ccc} T_pM & \xrightarrow{\mathrm{d}F_p} & T_{F(p)}N \\ \mathrm{d}\phi_g \downarrow \wr\| & & \|\wr \downarrow \mathrm{d}\psi_g \\ T_qM & \xrightarrow[\mathrm{d}F_q]{} & T_{F(q)}N \end{array}$$

  The vertical lines are isomorphisms thereby giving that $\mathrm{d}F_p$ and $\mathrm{d}F_q$ have the same rank. Since p and q are arbitrary, F has constant rank.

This applies directly to our scenario because if we take $G = SO(3,\mathbb{R})$, $M = SO(3,\mathbb{R})$, and $N = S^2$ we know that there is always a rotation taking $\overrightarrow{\pi(R)} \in S^2$ to $\overrightarrow{\pi(P)} \in S^2$. Furthermore, there is always a rotation that can take any $R \in SO(3,\mathbb{R})$ to $P \in SO(3,\mathbb{R})$. Thus, $\pi : SO(3,\mathbb{R}) \to S^2$ has constant rank. Now we want to show that a smooth surjective map of constant rank is submersion.

- Assume we have $F : M \to N$ smooth and constant rank. Suppose F is not a submersion, implying that $\text{rank}(F) = k < n = \dim(N)$. The rank theorem lets us know that at each point there is a smooth coordinate neighborhood in which F obtains the representation:

$$F(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

For any open cover we can pick countably many smooth charts $\{(U_i, \phi_i)\}$ for M and corresponding charts $\{(V_i, \psi_i)\}$ for N s.t. the sets $\{U_i\}$ cover M and F maps U_i into V_i. Now using the fact that any $k < n$ dimensional space will be a measure zero set in $\mathbb{R}^n$ when using the Lebesgue measure implies $m(F(U_i)) = 0$. Consequently:

$$m\Big(\bigcup F(U_i)\Big) \leq \sum m(F(U_i)) = 0$$

telling us that $F(M) \subset N$ is a measure zero set. Therefore, F cannot be surjective. We have proven that "not submersion $\implies$ not surjective" which is logically equivalent to "surjective $\implies$ submersion" for a constant rank map.

The only thing left is to convince ourselves of the surjectivity of $F : SO(3,\mathbb{R}) \to S^2$. The Gram-Schmidt process, given the set $\{v_1, v_2, v_3\}$, spits out the vectors $\{e_1, e_2, e_3\}$ where:

$$\begin{aligned} u_1 &= v_1 \quad \text{and} \quad e_1 = \frac{u_1}{\|u_1\|} \\ u_2 &= v_2 - \text{proj}_{u_1}(v_2) \quad \text{and} \quad e_2 = \frac{u_2}{\|u_2\|} \\ u_3 &= v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3) \quad \text{and} \quad e_3 = \frac{u_3}{\|u_3\|} \end{aligned}$$

where the new set consists of orthonormal vectors that could be identified as the columns or rows of a rotation matrix. To achieve surjectivity we take $v_1 = (a,b,c) \in S^2$ and note that $e_1 = v_1$ where this can be the last column specifically since order will not matter. So the rest of the task rests on finding two non-zero vectors in $\mathbb{R}^3$, not necessarily normalized, that are linearly independent of v_1. Since this is always possible we can take our list of vectors, input them into the Gram Schmidt process, and have the process spit out the necessary vectors to write a rotational matrix associated to our point on S^2. Finally, surjectivity combined with the fact that F is constant rank tells us F is a submersion. Note that all of this logic can be extended to say $F : SO(n+1,\mathbb{R}) \to S^n$ is a submersion.

Problem 3. As one of the HWs the coordinate transition map $x \to \frac{1}{x}$ came up for switching from one chart to another in the standard affine charts of $\mathbb{RP}^1$. Instead of viewing this map passively as a change of coordinates, we can look at it actively, as a transformation of $F : \mathbb{RP}^1 \to \mathbb{RP}^1$, defined within the single affine chart $Y \neq 0$. Find the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ which induces the map $f(x) = \frac{1}{x}$ in this standard affine chart.

Solution 3. To determine such a transformation we just need to look at three points. First we have the extremes:

$$\mathcal{M}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}k_1\\0\end{pmatrix} \quad \text{and} \quad \mathcal{M}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\k_2\end{pmatrix}$$

which forces:

$$\mathcal{M} = \begin{pmatrix}0 & k_1\\ k_2 & 0\end{pmatrix}$$

Since we are looking to invert over the line $y = x$, we also need:

$$\mathcal{M}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}m\\m\end{pmatrix}$$

which actually forces $k_1 = k_2$. Furthermore, since any scalar multiple will do, the nicest choice is:

$$\mathcal{M} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$