MATH 208 - HW # 5

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Problem 1. Continuing with the notation of the previous two problems, so that $\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$ and $V(x) = x^2 \frac{\partial}{\partial x}$, show that the vector field $V(x)$ is complete when viewed as a vector field on $\mathbb{RP}^1$.

Solution 1. From the previous problems we know that the change of coordinates we want to perform is given by:

$$y = \frac{1}{x} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial}{\partial y} = -\frac{1}{x^2}\frac{\partial}{\partial y} = -y^2\frac{\partial}{\partial y}$$

Plugging this into our vector field provides:

$$V'(y) = V(y^{-1}) = y^{-2}\Big(-y^2\frac{\partial}{\partial y}\Big) = -\frac{\partial}{\partial y}$$

Thus, when viewed as a vector field on $\mathbb{RP}^1$, the solution curves take the form:

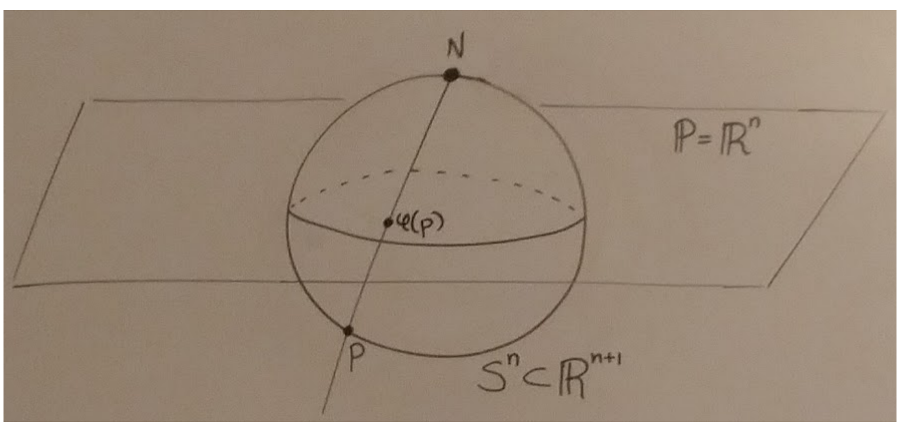
$$x(t) = -t + x_0$$

No solution blows up in finite time showing that the vector field is complete when viewed on $\mathbb{RP}^1$.

Problem 2. Compute stereo projection $\phi_N : S^2 \setminus \{N\} \to \mathbb{R}^2$ and its inverse.

Solution 2. Note that we take N to be the "North" pole, i.e. $N = (0, 0, \ldots, 0, 0, 1)$.

- First we will compute $\phi_N : S^n \setminus \{N\} \to R^n$ for arbitrary n. Our setup looks like:



where $p = (p_1, \ldots, p_{n+1}) \in S^n$, $\mathbb{P}$ is the plane of projection, and $\phi(p) \in \mathbb{P}$. Now to project from the n-sphere to $\mathbb{P}$ we first define the direction vector from N to p:

$$\overrightarrow{v} = p - N = (p_1, \ldots, p_n, p_{n+1} - 1)^T$$

and parametrize the line drawn as:

$$\overrightarrow{l}(t) = t\overrightarrow{v} + \overrightarrow{N} = (p_1 t, p_2 t, \ldots, p_n t, (p_{n+1} - 1)t + 1)^T$$

It is now our task to determine at what time the line intersects $\mathbb{P}$. This condition occurs when the last component vanishes, i.e. $(p_{n+1} - 1)t + 1 = 0$. Thus, the time when the line intersects $\mathbb{P}$ is $t' = \frac{1}{1 - p_{n+1}}$. So now we explicitly write the mapping as:

$$\phi_N : (p_1, \ldots, p_n, p_{n+1}) \to \Big(\frac{p_1}{1 - p_{n+1}}, \ldots, \frac{p_n}{1 - p_{n+1}}, 0\Big)$$

- We continue now to compute the inverse mapping. The picture from above still stands where we are now given $\phi(p) = q = (q_1, \ldots, q_n) \in \mathbb{P}$ and want to compute p. Again we define a direction from q to N:

$$\overrightarrow{w} = N - q = (-q_1, -q_2, \ldots, -q_n, 1)^T$$

and parametrize the line from q to N as:

$$\overrightarrow{r}(t) = t\overrightarrow{w} + \overrightarrow{q} = (-q_1 t + q_1, -q_2 t + q_2, \ldots, -q_n t + q_n, t)^T$$

It is now our task to determine at what time the line intersects S^n. This condition occurs when the components satisfy the equation:

$$\begin{aligned}
1 &= \sum_{i=1}^{n+1} x_i^2 \\
1 &= t^2 + \sum_{i=1}^{n} (-q_i t + q_i)^2 \\
1 &= t^2 + \sum_{i=1}^{n} (q_i^2 t^2 - 2q_i^2 t + q_i^2) \\
0 &= \left(\sum_{i=0}^{n} q_i^2 + 1 \right) t^2 - \left(2 \sum_{i=0}^{n} q_i^2 \right) t + \left(\sum_{i=0}^{n} q_i^2 - 1 \right)
\end{aligned}$$

Letting $\alpha = \sum_{i=1}^{n} q_i^2$ we find the solutions:

$$t'' = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha+1)(\alpha-1)}}{2(\alpha+1)} = 1, \frac{\alpha - 1}{\alpha + 1}$$

Naturally we do not accept $t'' = 1$ as a solution because that references the North pole. Using t'' we write down the inverse mapping explicitly:

$$\phi_N^{-1} : (q_1, \ldots, q_n, 0) \to \Big(\frac{2q_1}{\alpha+1}, \ldots, \frac{2q_n}{\alpha+1}, \frac{\alpha-1}{\alpha+1} \Big)$$

- Let us observe the special case of $n = 2$, which is what the problem is actually asking for. The mappings take the explicit forms:

$$\begin{aligned}
\phi_N &: S^2 \setminus \{N\} \to \mathbb{R}^2 \quad \text{given by} \quad \phi_N(x, y, z) = \Big(\frac{x}{1-z}, \frac{y}{1-z} \Big) \\
\phi_N^{-1} &: \mathbb{R}^2 \to S^2 \setminus \{N\} \quad \text{given by} \quad \phi_N^{-1}(a, b) = \Big(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1} \Big)
\end{aligned}$$

- For the case of $n = 2$ we can give another construction using Complex Analysis that draws inspiration from the form taken above. Define $\zeta = \frac{x+iy}{1-z}$ for a triplet in the forward direction and $\eta = a + ib$ for a pair in the backward direction. Notice:

$$\begin{aligned}
\phi_N &: S^2 \setminus \{N\} \to \mathbb{C} \quad \text{given by} \quad \phi_N(x, y, z) = \zeta \\
\phi_N^{-1} &: \mathbb{C} \to S^2 \setminus \{N\} \quad \text{given by} \quad \phi_N^{-1}(\eta) = \left(\frac{2\mathfrak{Re}(\eta)}{1+\overline{\eta}\eta}, \frac{2\mathfrak{Im}(\eta)}{1+\overline{\eta}\eta}, \frac{-1+\overline{\eta}\eta}{1+\overline{\eta}\eta} \right)
\end{aligned}$$

Problem 3. For any point $p \in S^2$ we have the stereographic projection $\phi_x : S^2 \backslash \{p\} \to x^{\perp}$. For $x = N$ and $x = S$ we have $x^{\perp} = \mathbb{R}^2$ where N and S denote the North and South poles respectively. Compute the overlap map $\phi_S \circ \phi_N^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

Solution 3. Let us generalize this to the case of $\phi_x : S^n \setminus \{N\} \to x^{\perp}$. Intuition naturally tells us that the overlap can be defined everywhere but the two poles. To see this explicitly:

$$(\phi_S \circ \phi_N^{-1})(0) = \phi_S(\phi_N^{-1}(0)) = \phi_S(S) = \infty$$
$$(\phi_S \circ \phi_N^{-1})(\infty) = \phi_S(\phi_N^{-1}(\infty)) = \phi_S(N) = 0$$

This is not surprising since an overlap map can only be defined on the intersection of the two charts. Besides this we can compute the explicit form of the composition using the results above. Take α as above and notice:

$$\phi_N^{-1} : (q_1, \ldots, q_n, 0) \to \Big(\frac{2q_1}{\alpha+1}, \ldots, \frac{2q_n}{\alpha+1}, \frac{\alpha-1}{\alpha+1}\Big)$$
$$\phi_S : (p_1, \ldots, p_n, p_{n+1}) \to \Big(\frac{p_1}{1+p_{n+1}}, \ldots, \frac{p_n}{1+p_{n+1}}, 0\Big)$$

provides:

$$\phi_S \circ \phi_N^{-1} : (q_1, \ldots, q_n) \to \left(\frac{\frac{2q_1}{\alpha+1}}{1+\frac{\alpha-1}{\alpha+1}}, \ldots, \frac{\frac{2q_n}{\alpha+1}}{1+\frac{\alpha-1}{\alpha+1}}\right) = \Big(\frac{q_1}{\alpha}, \ldots, \frac{q_n}{\alpha}\Big)$$