MATH 208 - HW # 4

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Problem 1. View x as the affine coordinate for $\mathbb{RP}^1$ so as to identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$. Show that the vector field $\frac{\partial}{\partial x}$ on the open set $\mathbb{R} \subset \mathbb{RP}^1$ extends to a smooth vector field on all of $\mathbb{RP}^1$. Does this vector field vanish at ∞?

Solution 1. We can use the transformation $x \to \frac{1}{x}$ to send $\mathbb{R}$ to $\mathbb{RP}^1$ by identifying $\frac{1}{0} = \infty$. As a result we can determine the form of the vector field on $\mathbb{RP}^1$ as:

$$\frac{\partial}{\partial y} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial}{\partial x} = -\frac{1}{x^2}\frac{\partial}{\partial x}$$

This defines a smooth vector field on $\mathbb{RP}^1$. Furthermore, as $x \to \infty$ we must have $\frac{\partial}{\partial y} \to 0$ implying the vector field vanishes as ∞.

Problem 2. Consider the vector field $V(x) = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$. Show that it is incomplete, by showing solutions blow up in finite time.

Solution 2. Any trajectory along this vector field must satisfy:

$$x'(t) = x^2(t)$$

for some initial condition $x(0) = x_0$. The general solution takes the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2$$

$$\int \frac{\mathrm{d}x}{x^2} = \int \mathrm{d}t$$

$$-\frac{1}{x} = t + C$$

$$x(t) = -\frac{1}{t+C}$$

Using the initial condition we obtain the particular solution:

$$x(t) = \frac{x_0}{1 - tx_0}$$

For $t = \frac{1}{x_0}$ we have $x(t)$ blow up in finite time.