MATH 208 - HW # 3

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Problem 1. Prove that $f : so(3,\mathbb{R}) \to gl(3,\mathbb{R})$ given by $f(x) = e^x$ is a smooth map and that its rank is 3 at the origin.

Solution 1.

- Any $A \in so(3,\mathbb{R})$ will take the form:

$$A = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} \quad \text{and} \quad A^2 = \begin{pmatrix} -x^2-y^2 & -yz & xz \\ -yz & -x^2-z^2 & -xy \\ xz & -xy & -y^2-z^2 \end{pmatrix}$$

If we let $\theta = \sqrt{x^2+y^2+z^2}$, then we obtain:

$$A^3 = -\theta^2 A, \quad A^4 = -\theta^2 A^2, \quad A^5 = \theta^4 A, \quad A^6 = \theta^4 A^2, \quad A^7 = -\theta^6 A, \quad A^8 = -\theta^6 A^2, \quad \ldots$$

and consequently:

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \frac{A^5}{5!} + \frac{A^6}{6!} + \frac{A^7}{7!} + \frac{A^8}{8!} + \ldots \\ &= I + A + \frac{A^2}{2!} - \frac{\theta^2 A}{3!} - \frac{\theta^2 A^2}{4!} + \frac{\theta^4 A}{5!} + \frac{\theta^4 A^2}{6!} - \frac{\theta^6 A}{7!} - \frac{\theta^6 A^2}{8!} + \ldots \\ &= I + \Big(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta^6}{7!} + \ldots\Big)A + \Big(\frac{1}{2!} - \frac{\theta^2}{4!} + \frac{\theta^4}{6!} - \frac{\theta^6}{8!} + \ldots\Big)A^2 \\ &= I + \frac{1}{\theta}\Big(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\Big)A + \frac{1}{\theta^2}\Big(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} + \ldots\Big)A^2 \\ &= I + \frac{\sin(\theta)}{\theta}A + \frac{\cos(\theta)-1}{\theta^2}A^2 \\ &= \begin{pmatrix} 1-\alpha(x^2+y^2) & \beta x - \alpha yz & \beta y + \alpha xz \\ -\beta x - \alpha yz & 1-\alpha(x^2+z^2) & \beta z - \alpha xy \\ \beta y + \alpha xz & -\beta z - \alpha xy & 1-\alpha(y^2+z^2) \end{pmatrix} \end{aligned}$$

where $\alpha = \frac{\cos(\theta)-1}{\theta^2}$ and $\beta = \frac{\sin(\theta)}{\theta}$. So if we take charts (U,ϕ) and (V,ψ) s.t.:

$$\phi : so(3,\mathbb{R}) \to \mathbb{R}^3 \quad \text{and} \quad \psi : gl(3,\mathbb{R}) \to \mathbb{R}^9$$

then the map $g : \mathbb{R}^3 \to \mathbb{R}^9$ defined by $g = \psi \circ f \circ \phi^{-1}$ takes the form:

$$g(x,y,z) = \begin{pmatrix} 1-\alpha(x^2+y^2) \\ \beta x - \alpha yz \\ \beta y + \alpha xz \\ -\beta x - \alpha yz \\ 1-\alpha(x^2+z^2) \\ \beta z - \alpha xy \\ \beta y + \alpha xz \\ -\beta z - \alpha xy \\ 1-\alpha(y^2+z^2) \end{pmatrix}$$

with the corresponding Jacobian:

$$J(g) = \begin{pmatrix} -2\alpha x & \beta & \alpha z & -\beta & -2\alpha x & -\alpha y & \alpha z & -\alpha y & 0 \\ -2\alpha y & -\alpha z & \beta & -\alpha z & 0 & -\alpha x & -\beta & -\alpha x & -2\alpha y \\ 0 & -\alpha y & \alpha x & -\alpha y & -2\alpha z & \beta & \alpha x & -\beta & -2\alpha z \end{pmatrix}$$

Evaluation at $x = y = z = 0$ tells us $\alpha\Big|_{(0,0,0)} = -\frac{1}{2}$ and $\beta\Big|_{(0,0,0)} = 1$. Using this information provides:

$$J(g)\Big|_{(0,0,0)} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

This shows that f has rank 3 at the origin. Furthermore, since every component of g is smooth so is g.

Problem 2. Prove that the image of f from the previous problem is $SO(3,\mathbb{R}) \subset gl(3,\mathbb{R})$.

Solution 2.

- $\forall\ A \in so(3,\mathbb{R})$ we have $\det(A) = 0$ and $\det(e^A) = e^{\text{tr}(A)} = e^0 = 1$.
- If $P = e^A$ for an $A \in so(3,\mathbb{R})$, then $P^{-1} = e^{-A} = e^{A^T} = (e^A)^T = P^T$ implying $e^A \in SO(3,\mathbb{R})$.