MATH 208 - HW # 3 - Corrections

Nathan Marianovsky
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Problem 1. Prove that the image of f from the previous problem is $SO(3,\mathbb{R}) \subset gl(3,\mathbb{R})$.

Solution 1.

- The following tells us that $\mathfrak{Im}(f) \subseteq SO(3,\mathbb{R})$:
  - $\forall\ A \in so(3,\mathbb{R})$ we have $\det(A) = 0$ and $\det(e^A) = e^{\text{tr}(A)} = e^0 = 1$.
  - If $P = e^A$ for an $A \in so(3,\mathbb{R})$, then $P^{-1} = e^{-A} = e^{A^T} = (e^A)^T = P^T$ implying $e^A \in SO(3,\mathbb{R})$.

- We also check that f is injective by determining the kernel, $\ker(f) = \{A \in so(3,\mathbb{R}) \mid f(A) = I\}$, of the group homomorphism:

$$\begin{aligned}
I &= f(A) \\
I &= I + \frac{\sin(\theta)}{\theta}A + \frac{\cos(\theta)-1}{\theta^2}A^2 \\
0 &= A\Big(\frac{\sin(\theta)}{\theta}I + \frac{\cos(\theta)-1}{\theta^2}A\Big) \\
A &= 0, \frac{\theta\sin(\theta)}{1-\cos(\theta)}I
\end{aligned}$$

  Since no multiple of the identity is skew-symmetric it must be that $A = 0$ showing that the kernel is trivial.

- Now we handle surjectivity:
  - To accomplish this we are going to need Rodrigues' Rotation Formula which states given a vector $v \in \mathbb{R}^3$ and a unit vector $\hat{k} \in \mathbb{R}^3$ describing an axis of rotation about which v rotates by an angle θ, according to the right hand rule the resulting vector after rotation is defined as:

$$v' = \cos(\theta)v + \sin(\theta)(\hat{k} \times v) + (1-\cos(\theta))(\hat{k}\cdot v)\hat{k}$$

    * Let us quickly derive this formula. To begin take the vector we want to rotate and split it into components relative to the axis $\hat{k}$:

$$v = v_\perp + v_\|$$

      where the parallel component is nothing more than:

$$v_\| = (v \cdot \hat{k})\hat{k}$$

      and the perpendicular component is:

$$v_\perp = v - v_\| = v - (\hat{k}\cdot v)\hat{k}$$

      We now want to use the known identity:

$$(a\cdot c)b - (a\cdot b)c = a \times (b \times c)$$

      to rewrite the perpendicular component as:

$$v_\perp = -\hat{k} \times (\hat{k} \times v)$$

      To continue we note that since we are rotating about the axis $\hat{k}$, the component parallel to it will not change under the rotation. Therefore, we have:

$$v'_\| = v_\|$$

The perpendicular component will transform according to:

$$\|v'_\perp\| = \|v_\perp\|$$
$$v'_\perp = \cos(\theta)v_\perp + \sin(\theta)(\hat{k} \times v_\perp)$$

which can be simplified because:

$$\hat{k} \times v_\perp = \hat{k} \times (v - v_\parallel) = \hat{k} \times v - \hat{k} \times v_\parallel = \hat{k} \times v$$

giving:

$$v'_\perp = \cos(\theta)v_\perp + \sin(\theta)(\hat{k} \times v)$$

We know that the above transformation preserves the norm because $v_\perp$ and $\hat{k} \times v$ have the same length. Now we can write down the explicit form of the rotated vector as:

$$\begin{aligned} v' &= v'_\parallel + v'_\perp \\ &= v_\parallel + \cos(\theta)v_\perp + \sin(\theta)(\hat{k} \times v) \\ &= v_\parallel + \cos(\theta)(v - v_\parallel) + \sin(\theta)(\hat{k} \times v) \\ &= \cos(\theta)v + (1 - \cos(\theta))v_\parallel + \sin(\theta)(\hat{k} \times v) \\ &= \cos(\theta)v + \sin(\theta)(\hat{k} \times v) + (1 - \cos(\theta))(\hat{k} \cdot v)\hat{k} \end{aligned}$$

– With the rotation formula in hand, we want to transform it into matrix mode. So we define the following:

$$K = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \quad \text{and} \quad K^2 = \begin{pmatrix} -k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & -k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & -k_x^2 - k_y^2 \end{pmatrix}$$

where:

$$Kv = \hat{k} \times v \quad \text{and} \quad K^2 v + v = (\hat{k} \cdot v)\hat{k}$$

allows us to rewrite the rotation formula as:

$$\begin{aligned} v' &= \cos(\theta)v + \sin(\theta)(\hat{k} \times v) + (1 - \cos(\theta))(\hat{k} \cdot v)\hat{k} \\ &= \cos(\theta)v + \sin(\theta)Kv + (1 - \cos(\theta))(K^2 v + v) \\ &= v + \sin(\theta)Kv + (1 - \cos(\theta))K^2 v \end{aligned}$$

This allows us to say that any $R \in SO(3, \mathbb{R})$ can be written down as:

$$R = I + \sin(\theta)K + (1 - \cos(\theta))K^2$$

– If we take R as above then we can always find a $K \in so(3, \mathbb{R})$ s.t. $e^K = R$ with $k_1^2 + k_2^2 + k_3^2 = 1$. This basically shows surjectivity, however we would like to get rid of the restriction on the components of K. In full generality we had:

$$e^A = I + \frac{\sin(\theta)}{\theta}A + \frac{(1 - \cos(\theta))}{\theta^2}A^2$$

Therefore, we can say that given a rotation matrix, R, we can always find an $A \in so(3, \mathbb{R})$ s.t.:

$$e^{\theta A} = R = I + \sin(\theta)A + (1 - \cos(\theta))A^2$$

which lifts the restriction on the components summing to one. Thus, $\mathfrak{Im}(f) \cong SO(3, \mathbb{R})$.