MATH 208 - HW # 2

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Problem 1. Prove that the cone, $x^2 + y^2 = z^2$ for $z \geq 0$, is a topological manifold, but is not a smooth embedded manifold in Euclidean 3-space.

Solution 1.

- To show that the cone, $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \text{ and } z \geq 0\}$, is a topological manifold we need to find a homeomorphism between $\mathbb{R}^n$ and $\mathcal{C}$ for some fixed n. The most natural choice is projection onto the xy-plane ($n = 2$) given by:
$$f(x, y, z) = (x, y) \quad \text{and} \quad f^{-1}(x, y) = (x, y, \sqrt{x^2 + y^2})$$
We check that f is a homeomorphism directly:
  - If $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$, then $(x_1, y_1) = (x_2, y_2)$ implying $x_1 = x_2$ and $y_1 = y_2$. Thus, f is injective.
  - For any $\overrightarrow{q} = (x, y) \in \mathbb{R}^2$ we can always find a unique point $\overrightarrow{p} = (x, y, \sqrt{x^2 + y^2}) \in \mathcal{C}$ s.t. $f(\overrightarrow{p}) = \overrightarrow{q}$. Thus, f is surjective.
  - Now using the fact that projection maps are continuous we know that f is continuous.
  - For f^{-1} we need to consider open sets in the topology endowed on the cone as a subspace of $\mathbb{R}^3$. Any such open set takes the form $U = \mathcal{C} \bigcap B_r(\overrightarrow{p})$ where $B_r(\overrightarrow{p})$ is an open ball in $\mathbb{R}^3$. Therefore, the open sets in this topology correspond exactly to the level curves of the cone. So taking any level curve $U \subseteq \mathcal{C}$ provides $f^{-1}(U) = \mathcal{B}_\rho(0)$ for $\mathcal{B}_\rho(0) \subseteq \mathbb{R}^2$ open and $\rho = \sqrt{x^2 + y^2}$.

- To see that $\mathcal{C}$ is not a smooth manifold we need to show that there is weird behavior occuring in the derivative at the origin. The gradient corresponding to our surface takes the form:
$$\nabla f = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \\ -1 \end{pmatrix}$$
To show that the gradient does not behave nicely at the origin we observe the value it decides to take on depending on the direction from which we approach. By letting $y = 0$ we arrive at:
$$\nabla f = \begin{pmatrix} \text{sgn}(x) \\ 0 \\ -1 \end{pmatrix}$$
Knowing that $\text{sgn}(x) \to -1$ if $x \to 0^-$ and $\text{sgn}(x) \to 1$ if $x \to 0^+$ we arrive at the conclusion that ∇f cannot be defined at the origin. Consequently, there does not exist a diffeomorphism between $\mathcal{C}$ and $\mathbb{R}^2$ implying $\mathcal{C}$ is not a smooth manifold.

Problem 2. Show that $u = xy$ and $v = y$ is not a good change of coordinates near the origin, while on the other hand $u = (x + .005)(y + .001)$ and $v = y$ is a good change of coordinates near the origin.

Solution 2.

- The first change of coordinates can be characterized as:

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{given by} \quad f(x, y) = \begin{pmatrix} xy \\ y \end{pmatrix}$$

  Where the Jacobian takes the form:

$$J(f) = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

  As a result, $\det(J(f)) = y$. Therefore, the Jacobian is not of full rank along the x-axis. So for any neighborhood of the origin, $\mathcal{B}_r(0)$, we will always have the Jacobian non-invertible. By the Inverse Function Theorem we can say that the change of coordinates cannot be inverted at the origin.

- The second change of coordinates can be characterized as:

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{given by} \quad g(x, y) = \begin{pmatrix} (x + .005)(y + .001) \\ y \end{pmatrix}$$

  Where the Jacobian takes the form:

$$J(g) = \begin{pmatrix} y + .001 & x + .005 \\ 0 & 1 \end{pmatrix}$$

  As a result, $\det(J(g)) = y + .001$. Therefore, the Jacobian is not of full rank along a copy of the x-axis at height $y = -.001$. By the Inverse Function Theorem we can say that g^{-1} exists in a neighborhood of the origin that does contain any of the bad points mentioned right before. For example you can consider $\mathcal{B}_{.0005}(0)$ to be such a neighborhood.