Geometry and Topology Preliminary Examination. UC Santa Cruz. Fall 2017

1. Recall that two submanifolds X and Y of a manifold M intersect transversely if for any $x \in X \cap Y$ we have $T_xX + T_xY = T_xM$. Let X be a submanifold of $M = \mathbb{R} \times P$, where P is a smooth manifold, and $\pi \colon M \to \mathbb{R}$ be the projection to the first component. Prove that X and the slice $\{t\} \times P$ intersect transversely if and only if t is a regular value of the function $\pi\mid_X$.

2. Let $F \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that F is necessarily orientation preserving at its regular points, i.e., $F^*(dx \wedge dy) = f dx \wedge dy$ with $f > 0$.

3. Let ω be an n-form on the n-dimensional manifold M. Assume that $\omega_p \neq 0$ at some point $p \in M$. Show that there exist local coordinates $x_1, \ldots, x_n$ near p such that $\omega = dx_1 \wedge \ldots \wedge dx_n$.

4. Let $K \subset \mathbb{R}^3$ be a cube made of wire, which is to say the union of the vertices and edges of the unit cube.

(1) What is the fundamental groups of K?

(2) Thicken K a bit, forming a smooth three-dimensional manifold whose boundary is the smooth surface X. Thus X is the set of all points a distance ε from K. (Smooth the corners of X if neccessary. Any $\varepsilon < 1/2$ works.) What are the homology groups of X?

5. Show that the integral homology groups H_i of a closed orientable simply connected 4-manifold are $H_0 \cong H_4 \cong \mathbb{Z}$, $H_1 = H_3 = 0$, and $H_2 \cong \mathbb{Z}^r$, a free abelian group of some rank $r \geq 0$.

6. Prove or disprove: $SO(3)$ admits a metric of constant sectional curvature.

7. View $x \in \mathbb{R}$ as the affine coordinate for $\mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}$ and let y be the 'other' affine coordinate centered at ∞.

A) Find the coordinate transition map relating x and y.

B) Express the translation vector field $\frac{\partial}{\partial x}$ on the line $\mathbb{R}$ in terms of the y- coordinates at infinity.

8. Let E_1, E_2, E_3 be pointwise linearly independent vector fields on some manifold and suppose that $[E_1, E_2] = E_3$. Find necessary and sufficient conditions for functions f, g so as to insure that $[fE_1, gE_2] = E_3$.