Geometry and Topology Prelim, UCSC, Fall 2016

1. Consider the vector fields
$$v = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - z^m\frac{\partial}{\partial z}$$
and
$$w = \frac{\partial}{\partial x} + z\frac{\partial}{\partial z}.$$
on $\mathbb{R}^3$, where $m \geq 0$.

(a) Find the (local) flow of v. Is v complete? (Note: Pay attention to m.)

(b) Find the bracket $[v, w]$.

(c) Is the distribution spanned by v and w (on the open subset where these vector fields are linearly independent) integrable?

2. Let α and β be differential forms on a manifold M and let X be a vector field. Prove that
$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta).$$

3. Is there an embedding of the Klein bottle in $\mathbb{R}^4$? Construct such an embedding or prove that it does not exist.

4. Consider the special orthogonal group $\mathrm{SO}(n)$, i.e., $\mathrm{SO}(n)$ is formed by $n \times n$ matrices A with $AA^T = I$ and $\det A = 1$.

(a) Show that $\mathrm{SO}(n)$ is a smooth manifold.

(b) Show that $\mathrm{SO}(n)$ is compact.

(c) Is the tangent bundle $T\,\mathrm{SO}(n)$ trivial? Justify your answer.

5. Let $(G, *)$ be a topological group with multiplication $*\colon G \times G \to G$. In G, there are two ways to multiply loops based at the identity element $e \in G$:

(i) Use the usual concatenation of loops,
$$(f \cdot g)(s) = \begin{cases} f(2s), & 0 \leq s \leq 1/2, \\ g(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

(ii) Use the group structure in G:
$$(f * g)(s) = f(s) * g(s), \qquad 0 \leq s \leq 1.$$

Show that $f \cdot g \simeq f * g$ for all loops f and g based at e. Thus, two ways of multiplying loops define the same product structure in $\pi_1(G, e)$. Next show that $\pi_1(G, e)$ is in fact abelian regardless of whether G is abelian or not.

Over please!

6. Let X be the topological space obtained by attaching the boundary ∂D of the closed unit disc D to the unit circle via a map $\varphi\colon \partial D = S^1 \to S^1$ of degree three. (For instance, we can take $\varphi(z) = z^3$, where we view S^1 as the unit circle in $\mathbb{C}$.) Find the homology of X with integer coefficients. What changes if we use rational coefficients?

7. Consider the upper half-plane $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Riemannian metric

$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

(a) Find the equation $t \mapsto (x(t), y(t))$ of the geodesic starting at $(0, 1)$ with initial velocity $(0, 1)$. (Hints: You can use the fact that the Christoffel symbols of g are zero except for $\Gamma^y_{xx} = 1/y$ and $\Gamma^x_{xy} = \Gamma^y_{yx} = \Gamma^y_{yy} = -1/y$. You don't need to calculate them. Alternatively, and this is a better method, you can base your argument on the observation that the metric is invariant with respect to the reflection in the y-axis.)

(b) What is the distance between the points $(0, a)$ and $(0, b)$?