Geometry and Topology Preliminary Examination   **Fall 2015**

Instructions: Answer seven of the following eight problems. Indicate clearly which problems you want graded—we won't pick your best ones for you.

1.   2.   3.   4.   5.   6.   7.   8.

1. Prove or provide a counter-example: Any integral manifold of a smooth vector field on a manifold M is an embedded submanifold of M.

2. Let M be an n-dimensional Riemannian manifold. Define the *unit tangent bundle* $UT(M)$ of M by
$$UT(M) := \coprod_{m \in M} \{v_m \in T_m M : ||v_m||_m = 1\}.$$
Show that $UT(M)$ is a $2n-1$ dimensional embedded submanifold of TM.

3. (a) Let X be a smooth vector field on a smooth manifold M. Prove or provide a counter-example: If β is a closed form on M, then the Lie derivative $\mathcal{L}_X \beta$ of β is exact.

   (b) Find the Lie derivative of the two-form $\beta(x, y) = (x^k + y^k)\, dx \wedge dy$ on $\mathbb{R}^2$ with respect to the vector field $X(x, y) = x^\ell \partial_y$ for $k, \ell \in \mathbb{N}$.

4. Show that if A is a retract of X, then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

5. Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, and G is transverse to an embedded submanifold $X \subseteq P$. Show that F is transverse to $G^{-1}(X)$ if and only if $G \circ F$ is transverse to X.

6. A polynomial $p : \mathbb{C} \to \mathbb{C}$ can be extended to a continuous map $\hat{p} : S^2 \to S^2$ of one-point compactifications. Show that the degree of $\hat{p}$ equals the degree of p as a polynomial.

7. A connection ∇ on the tangent bundle TM of a smooth manifold M induces a connection ∇^* on the cotangent bundle T^*M by requiring
$$X(\iota_Y \alpha) = \iota_Y \nabla^* X \alpha + \iota_{\nabla_X Y} \alpha$$
for all smooth vector fields X and Y, and all one-forms α. Show that the condition that ∇ is torsion free is equivalent to the condition that for any one-form α and smooth vector fields X and Y we have
$$d\alpha(X, Y) = \nabla^*_X \alpha(Y) - \nabla^*_Y \alpha(X).$$

8. Prove or provide a counter-example: If a Lie group G acts transitively on a smooth manifold M, then any G-equivariant vector field on M is complete.