Geometry and Topology Preliminary Exam, Fall 2014

[1] Which of the following manifolds are diffeomorphic and which are not:
(a) $\mathbb{R}P^2$, $\mathbb{C}P^1$, and S^2.
(b) $\mathbb{R}P^3$, S^3, SO(3), SU(2), and the unit tangent bundle to S^2.
Justify your conclusions.

[2] Consider the following two vector fields v, w on the plane

$$v = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}, \qquad w = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

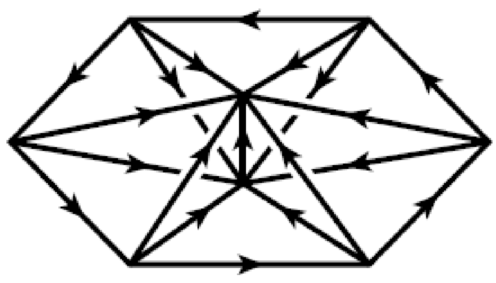
(a) Are these vector fields complete?
(b) Find the flow of v.
(c) Find the bracket $[v, w]$.

[3] Consider the map of wedge product $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ which sends v, w to their wedge product $v \wedge w$.
(i) What are the critical points of ϕ?
(ii) What are the critical values of ϕ?
(iii) What is the dimension of the image of ϕ?
(iv) What is the image of ϕ?

[4] Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ be the cylinder of unit radius.
(a) Show that the geodesics on M are the helices, that is, curves which cut each generator (= each vertical line) at the same angle (or have a constant angle with the z-axis), the generators themselves, and the circles of intersection with planes z = constant.
(b) How many geodesics connect two given points p, q on M?
(c) Show that a geodesic starting at a point (x, y, z) in M does not minimize arc length after it passes through the antipodal line $\{(-x, -y, t) \mid t \in \mathbb{R}\}$.

[5] On a closed orientable surface Σ_{g+h} of genus $g + h$ with $g, h \geq 0$, let C be a loop that separates Σ_{g+h} into two compact surfaces $\Sigma'_g = \Sigma_g - \{\text{open disc}\}$ and $\Sigma'_h = \Sigma_h - \{\text{open disc}\}$ of genus g and h, respectively. Show that Σ'_g does not retract onto its boundary C, and hence Σ_{g+h} does not retract onto C.

[6] Construct a 3-dimensional Δ-complex X from four oriented tetrahedra T_1, T_2, T_3, T_4 by the following two steps. (The picture below uses six tetrahedra. Here we use **four** for simplicity.) First arrange the tetrahedra in a cyclic pattern so that all four tetrahedra share the same single edge, and each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts taken mod 4. Then identify the bottom face of T_i with the top face of T_{i+1} by orientation preserving homeomorphism for each i. Show that the homology of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}_4, 0, \mathbb{Z}$.



[7] Show that if M is a compact smooth orientable surface in $\mathbb{R}^3$ that is not diffeomorphic to a sphere, then there is a point p in M at which the Gaussian curvature is negative.