(with the identity map as a global trivialization). If M is a smooth manifold, then $M \times \mathbb{R}^k$ is smoothly trivial.

Although there are many vector bundles that are not trivial, the only one that is easy to visualize is the following.

Example 5.2 (The Möbius Bundle). Let $I = [0,1] \subset \mathbb{R}$ be the unit interval, and let $p\colon I \to \mathbb{S}^1$ be the quotient map $p(x) = e^{2\pi i x}$, which identifies the two endpoints of I. Consider the "infinite strip" $I \times \mathbb{R}$, and let $\pi_1\colon I \times \mathbb{R} \to I$ be the projection on the first factor. Let $\sim$ be the equivalence relation on $I \times \mathbb{R}$ that identifies each point $(0,y)$ in the fiber over 0 with the point $(1,-y)$ in the fiber over 1; in other words, the right-hand edge is given a half-twist to turn it upside-down, and then is glued to the left-hand edge. Let $E = (I \times \mathbb{R})/{\sim}$ denote the resulting quotient space, and let $q\colon I \times \mathbb{R} \to E$ be the quotient map (Figure 5.2).

Because $p \circ \pi_1$ is constant on each equivalence class, it descends to a continuous map $\pi\colon E \to \mathbb{S}^1$. A straightforward (if tedious) verification shows that this makes E into a smooth real line bundle over $\mathbb{S}^1$, called the *Möbius bundle*. (One local trivialization of E is obtained in an obvious way from the restriction of the identity map to $(0,1) \times \mathbb{R}$, which descends to the quotient to yield a homeomorphism from $\pi^{-1}(\mathbb{S}^1 \smallsetminus \{1\})$ to $(0,1) \times \mathbb{R}$. It takes a bit more work to construct a local trivialization whose domain includes the fiber where the gluing took place. Once this is done, the two local trivializations can be interpreted as coordinate charts defining the smooth structure on E. Problem 5-2 asks you to work out the details. Later in the book, Problem 9-18 will suggest a more powerful approach.) For any $r > 0$, the image under q of the rectangle $I \times [-r,r]$ is a smooth

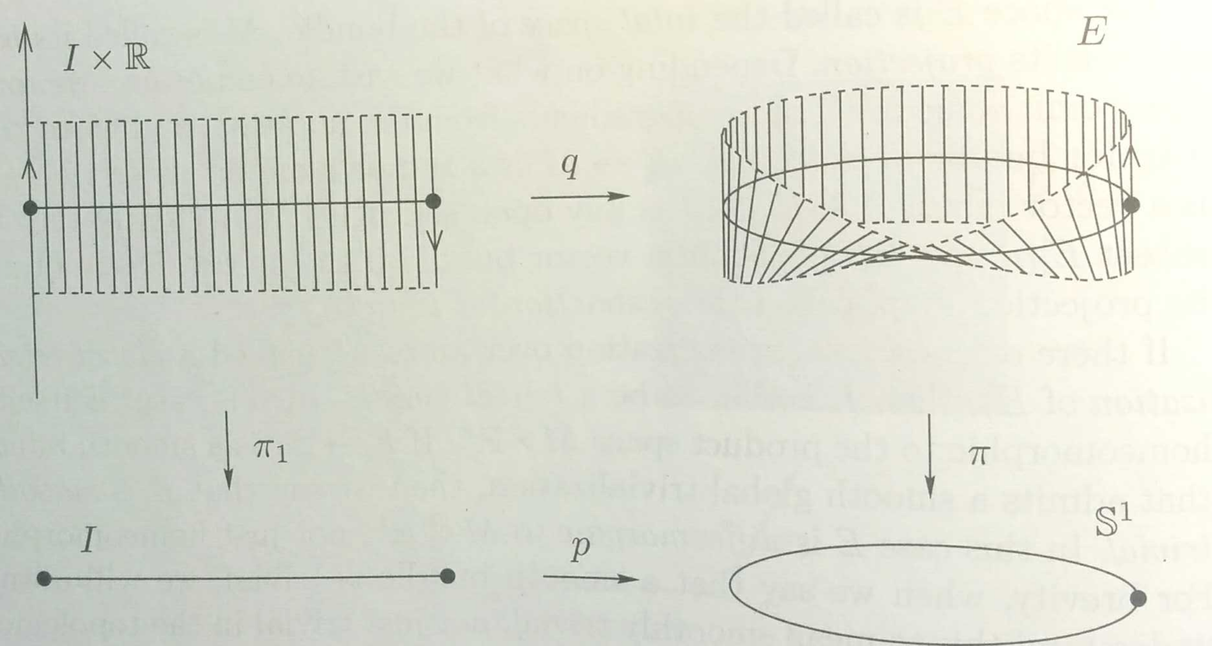

Figure 5.2. (Part of) the Möbius bundle.

compact manifold with boundary called the *Möbius band*; you can make a paper model of this space by gluing the ends of a strip of paper together with a half-twist.

The most important examples of vector bundles are tangent bundles of smooth manifolds.

Proposition 5.3 (The Tangent Bundle as a Vector Bundle). *Let M be a smooth n-manifold and let TM be its tangent bundle. With its standard projection map, its natural ...*