Then there is a unique isomorphism $\Phi\colon V \otimes W \to Z$ such that $\Phi \circ \pi$, where $\pi\colon V \times W \to V \otimes W$ is the canonical projection. [Remark: This shows that the details of the construction used to define the tensor product are irrelevant, as long as the resulting space satisfies the characteristic property.]

11-2. If V is any finite-dimensional real vector space, prove that there are canonical isomorphisms $\mathbb{R} \otimes V \cong V \cong V \otimes \mathbb{R}$.

11-3. Let V and W be finite-dimensional real vector spaces. Prove that there is a canonical (basis-independent) isomorphism between $V^* \otimes W$ and the space $\operatorname{Hom}(V, W)$ of linear maps from V to W.

11-4. Let M be a smooth n-manifold, and let σ be a smooth covariant k-tensor field on M. If $(U, (x^i))$ and $(\widetilde{U}, (\widetilde{x}^j))$ are overlapping smooth charts on M, we can write

$$\sigma = \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} = \widetilde{\sigma}_{j_1 \dots j_k} d\widetilde{x}^{j_1} \otimes \cdots \otimes d\widetilde{x}^{j_k}.$$

Compute a transformation law analogous to (6.7) expressing the component functions $\sigma_{i_1 \dots i_k}$ in terms of $\widetilde{\sigma}_{j_1 \dots j_k}$.

11-5. Generalize the coordinate transformation law of Problem 11-4 to mixed tensors of any rank.

11-6. Suppose $F\colon M \to N$ is a diffeomorphism. For any pair of nonnegative integers k, l, show that there are smooth bundle isomorphisms $F_*\colon T^k_l M \to T^k_l N$ and $F^*\colon T^k_l N \to T^k_l M$ satisfying

$$\begin{aligned} F_* S\left(X_1, \dots, X_k, \omega^1, \dots, \omega^l\right) &= S\left(F_*^{-1} X_1, \dots, F_*^{-1} X_k, F^*\omega^1, \dots, F^*\omega^l\right), \\ F^* S\left(X_1, \dots, X_k, \omega^1, \dots, \omega^l\right) &= S\left(F_* X_1, \dots, F_* X_k, F^{-1*}\omega^1, \dots, F^{-1*}\omega^l\right). \end{aligned}$$

11-7. Let M be a smooth manifold.

(a) Given a smooth covariant k-tensor field $\tau \in \mathcal{T}^k(M)$, show that the map $\mathcal{T}(M) \times \cdots \times \mathcal{T}(M) \to C^\infty(M)$ defined by

$$(X_1, \dots, X_k) \mapsto \tau(X_1, \dots, X_k)$$