PROBLEMS FROM LEE:

on Vector Bundles: chapter 5, problems 5-2, 5-5, 5-12.

on Vector fields and parallelization : 5-10. (Here you are asked to check some details around a parallelization of S^3. See 'My PROBLEM 1" for where these three vector fields came from.)

on Tensor Fields: 11-4.

MY PROBLEMS

ON PARALLELIZATION and the 3-sphere as a group.

My PROBLEM 1. Identify S^3 with $SU(2)$ or- what is the same – $Sp(1)$ - the unit quaternions - in the standard way by letting $SU(2)$ act on S^3 and observing that the action is free and transitive. Write down this diffeo explicitly. Show that the vector fields of Lee's problem 5-10 correspond to the standard basis for the Lie algebra, namely i, j, k in quaternionic language. [See also Lee's problem 9-9, 8-19 and 8-20]

ON THE NORMAL BUNDLE; TUBES, etc.

Background.

The *normal bundle* $N(C)$ to a smooth embedded submanifold C in Euclidean space $\mathbb{R}^n$ is the vector bundle whose fiber at $p \in C$ consists of all vectors orthogonal ("normal") to the tangent space T_pC. Thus $N(C) = \{(p, v) \in C \times \mathbb{R}^n : v \perp T_pC\}$. The *normal bundle map*, or "exponential map" is the map $N(C) \to \mathbb{R}^n$ sending (p, v) to $p + v$.

MY PROBLEM 2. Let C be the parabola $y = x^2$ in the plane $\mathbb{R}^2$. Compute the locus of critical values of the normal bundle map $N(C) \to \mathbb{R}^2$.

Hint: this locus will be a cubic cusp, lying above the parabola, having the reflectional symmetry of the parabola.

Hint 2 - if you are stuck go to problem 2.

Background. Take the special case of planar closed curves, so that C is a closed smooth embedded curve of length L in the plane $\mathbb{R}^2$. Its points can be parameterized by arc length via a map $s \mapsto c(s) \in \mathbb{R}^2$ from $\mathbb{R} \to \mathbb{R}^2$ whose image is C and which is periodic of period L: $c(s+L) = c(s)$. Write $T(s) = d\gamma/ds$ for the corresponding unit tangent vector to C. Let J be the operation of rotation by 90 degrees counterclockise in the plane: $J(a, b) = (b, -a)$. Write $N(s) = JT(s)$ be its normal vector.

MY PROBLEM 3.

a) Show that the normal bundle to C is diffeomorphic to $S^1_L \times \mathbb{R}$ by sending $(s, t) \mapsto (c(s), tT(s))$. Here the circle S^1_L is the circle of "circumference" L, which is to say the quotient space $S^1 = \mathbb{R}/L\mathbb{Z}$. Then the normal exponential in s, t coordinates is the map $F(s, t) = \gamma(s) + tT(s)$.

b) Show this map F is a diffeomorphism from a neighborhood of $S^1 \times \{0\} \subset S^1 \times \mathbb{R}$ to a neighborhood of C in the plane.

c) For $p = F(s, t)$ in this diffeomorphic neighborhood of (b), show that the distance from p to C is $|t|$ and the closest point to p on the curve C is $c(s)$.

d) Relate the set of critical values (s_*, t_*) of the normal exponential map to the curvature at $c(s_*)$ of the curve.

Hints. Use T, N as basis in the range space for your computations of the rank of dF. Look up the 2-dimensional Frenet-Serret equations if you have forgotten them.

MY PROBLEM 4. Repeat, problem 2, a closed space curve. Thus, suppose now that $c : \mathbb{R} \to \mathbb{R}^3$ is a smooth periodic immersion of period L as before, so that c defines an embedded closed curve of length L, and with $\|dc/ds(s)\| = 1$. Setting $T(s) = dc/ds$, the unit tangent, we define the normal $N(s)$ by $dT/ds = \kappa(s)N(s)$, the first of the Frenet-Serret equations, and we assume, for simplicity that $\kappa(s)$ never vanishes. Define the binormal $B(s) = T(s) \times N(s)$ [vector cross product] , so that $T(s), N(s), B(s)$ is a moving frame along the curve the "Frenet-Serret frame". Set $F(s, u, v) = c(s) + uN(s) + vB(s)$.

a) Show F is a diffeomorphism of a neighborhood of $S^1_L \times \{0\}$ in $S^1_L \times \mathbb{R}^2$ to $\mathbb{R}^3$.

b) Formulate and prove the analogue of Problem 3 b.

c) Show that for ϵ sufficiently small, let $S^1(\epsilon)$ denote the radius ϵ circle in the uv plane. Show that the image under F of $S^1_L \times S^1(\epsilon)$ consists of the points of $\mathbb{R}^3$ whose distance from the image curve is precisely ϵ. Show this space is an embedded torus.

d) Find conditions on curvature κ and torsion τ which guarantee that F becomes singular at point (s, u, v) with $u(\kappa(s), \tau(s)), v(\kappa(s), \tau(s))$ for some functions u, v that you are to determine.

PROBLEM 5. For any embedded submanifold C of $\mathbb{R}^n$ show that $TC \oplus N(C)$ is the trivial bundle, $C \times \mathbb{R}^n$, when viewed as a smooth vector bundle over C.

ON QUOTIENT SPACES

MY PROBLEM 6.

Define an action of the integers $\mathbb{Z}$ on $\mathbb{R}^3 \setminus \{0\}$ by $k \cdot (x, y, z) = 2^k(x, y, z) := (2^k x, 2^k y, 2^k z)$ for k an integer. Prove that the corresponding quotient space $(\mathbb{R}^3 \setminus \{0\})/\mathbb{Z}$ is diffeomorphic to $S^2 \times S^1$.

Hint: rewrite this $\mathbb{Z}$-action in spherical coordinates.