Deriving the geodesics.

Showing lines are intersections of planes w surface
All three geometries: realized as surfaces $S$ in 3-space $\mathbb{R}^3$, parameterized in polar coord $(r, \theta)$ as:

$$\vec{X}(r, \theta) = g(r)\vec{U}_3 + f(r)(\cos(\theta)\vec{U}_1 + \sin(\theta)\vec{U}_2).$$

With $f'(r) = g(r)$, $f(0) = 0, g(0) = 1$, $\vec{U}_1, \vec{U}_2, \vec{U}_3$ a basis of vectors in $\mathbb{R}^3$. So $\vec{X}(0, \theta) = \vec{U}_3$.

Euclidean: $\vec{U}_1, \vec{U}_2$ E.- orthonormal. $f(r) = r$. $g(r) = 1$

Spherical: $\vec{U}_1, \vec{U}_2, \vec{U}_3$ E.- orthonormal. $f(r) = \sin(r)$. $g(r) = \cos(r)$.

Hyperbolic: $\vec{U}_1, \vec{U}_2, \vec{U}_3$ M.- orthonormal. $f(r) = \sinh(r)$. $g(r) = \cosh(r)$. $\vec{U}_3$ of M.-length $-1$. 
Arclength squared: $ds^2 = dr^2 + f(r)^2 d\theta^2$.

Curves leaving $U_3$, lying on surface: parameterize $r(t), \theta(t)$ by auxiliary parameter $t$, $0 \leq t \leq T$. $r(0) = 0$.

Length of a curve $c$: $\ell(c) = \int_c ds := \int_0^T \sqrt{(\frac{dr}{dt}(t))^2 + f(r(t))^2 (\frac{d\theta}{dt}(t))^2} dt$
Def. A minimizing geod from $P = \vec{U}_3$ to $Q$ on surface is a curve on the surface which joints $P$ to $Q$ and minimizes $\ell(c)$ among all such curves.

Thm. The min. geodesics from P to Q are the curves $\theta = \theta_0$, $r = t$, $0 \leq t \leq r_0$ where $Q = \vec{X}(r_0, \theta_0)$. The length of this curve is $r_0 = d(P, Q)$.

Pf. Let $a, b$ be arb real numbers. Then :

$$a^2 + b^2 \geq a^2 \text{ with } "=\text{" iff } b = 0$$

so $\sqrt{a^2 + b^2} \geq \sqrt{a^2} = |a|$ with “=” iff $b = 0$
so : \[ \sqrt{(\frac{dr}{dt}(t))^2 + f(r(t))^2 (\frac{d\theta}{dt}(t))^2} \geq |(\frac{dr}{dt}(t)| \] with “=” iff \( \frac{d\theta}{dt} = 0 \).

This is true for each value of \( t \). Integrating this inequality :

\[ \int_c ds \geq \int_0^T |\frac{dr}{dt}(t)| dt \] with “=” iff \( \frac{d\theta}{dt} \equiv 0 \).

Now \( \int_0^T |\frac{dr}{dt}(t)| dt \geq \int_0^T \frac{dr}{dt}(t)| dt = \int dr = r_0 = \ell(c_0) \). Where \( c_0 \) is the curve \( \theta = \theta_0, 0 \leq r \leq r_0 \) described in the theorem.

This proves that \( \ell(c) \geq \ell(c_0) \) with equality if and only if \( c = c_0 \) (assuming all curves differentiable), proving the theorem. QED
In terms of our param of $S$ as surface in $\mathbb{R}^3$ the curve $c_0$ is $r \mapsto \vec{X}(r, \theta_0) = g(r)\vec{U}_3 + f(r)\vec{B}_3$ where $\vec{B}_3 = \cos(\theta_0)\vec{U}_1 + \sin(\theta_0)\vec{U}_2$ is a (unit) constant vector in space.

Now consider the linear subspace $L$ (plane through origin) spanned by $\vec{B}_3$ and $\vec{U}_3$

Exer: $L \cap S$ is the curve $c_0$ just described.

Cor. In all three cases the geodesics thru $P \in S$ are the intersections of $S$ with planes $L$ through the origin containing $P$. 