POINCARÉ’s MODELS

of the

HYPERBOLIC PLANE
‘Universe’ = Open Region of the usual plane $\mathbb{C}$ = model of hyperbolic plane

Boundary of region = $\partial$ = “the Ideal” = points at $\infty$

Geodesics = “Circles” perpendicular to $\partial$

Orientation preserving Isometries = LINEAR FRACTIONAL TRANSFORMATIONS WHICH map the Universe onto itself. NAME: $ISOM_+$

Angles = What they look like! (as per Euclidean). Model is “Conformal”

Distances: any point in the Universe has an infinite distance from the ideal
Upper half plane: Region: $y > 0$

$\partial = \text{Ideal} = \{y = 0\} = \text{real axis}$.

Geodesics = vertical (Euc.) rays $x = x_0, y > 0$. And (Euc.) half-circles w center on ideal ,

$ISOM_+ =$ Linear fractionals with $a, b, c, d$ real , $ad - bc > 0$. NAME: $\mathbb{PGL}(2, \mathbb{R})$ or $Sl(2, \mathbb{R})/\pm 1$

Angles = what they look like!

Infinitesimal distance: $ds = \frac{|dz|}{y}, |dz| = \sqrt{dx^2 + dy^2} = ds_{Euc}$
DISC MODEL: Region = unit disc $r < 1$ where $r = |z| = \sqrt{x^2 + y^2}$

$\partial = \{r = 1\} =$ unit circle.

Geodesics = Diameters. And arcs of circles perpendicular to $\partial$

$ISOM_+ =$ Linear fractionals whose matrix preserves the Hermitian form $z_1 \bar{z}_1 - z_2 \bar{z}_2$. NAME: $PU(1, 1)$ or $SU(1, 1)/\pm 1$.

Angles = What they look like!

Infinitesimal distance $= ds = \frac{|dz|}{(1-r^2)}$, $|dz| = \sqrt{dx^2 + dy^2} = ds_{Euc}$
EXERCISE (SOME VERSION OF WHICH WILL LIKELY BE ON FINAL)

In each model draw a line. Draw a point not on that line. Draw lots and lots of parallels to the line through the point.
Following Klein’s Erlangen program, we first focus more intently on the transformation groups $ISO_M^+$ of these models -the rigid motions of the geometry – rather than their lines, angles, trigonometry, areas, parallels.

so ...

GO TO MOBIUS TRANSFORMATIONS slides and STEREO PROJ