TRANSFORMATIONS of the Riemann Sphere =

Linear Fractional (Möbius) Transformations.
PROPERTIES:

1) map ‘circles’ to ‘circles’:

   remember: lines = circles that happen to pass thru $z = \infty$ !

2) preserve angles

3) represented by two-by-two complex matrices

4) used to represent rigid motions in ALL THREE geometries!
Form of linear fractional: \( F(z) = \frac{az + b}{cz + d} \).

Special case: \( F(z) = az + b \). Do you see that this special case maps circles to circles? EUCLIDEAN CASE!

WE don’t yet know that \( z \mapsto 1/z \) maps circles to circles.

But suppose we did know ...
**Theorem:** Every Mobius transformation map circles to circles.

**PROOF:** supposing the previous slide, in particular that $G(z) = 1/z$ mapped circles to circles, then:

**EXERCISE:** Show that every Mobius transformation is a composition of maps of the form $F$ and the map $G$. 
Next: Showing that \( z \mapsto 1/z \) maps circles to circles.

A) What is a circle on the sphere?

B) Stereographic projection \( ST : S^2 \rightarrow \mathbb{C} \cup \{\infty\} \) takes circles to ‘circles’ DERIVING STEREO PROJ...: IN CLASS!

C) Rotations of the sphere map circles to circles.

D) \( G(z) = 1/z \) is the composition \( ST \circ R_{\vec{e}_1,\pi} \circ ST^{-1} \) where \( R_{\vec{e}_1,\pi} \) = rotation by 180 degrees about the axis \( \vec{e}_1 = (1,0,0) \) and hence \( G \) maps circles to circles by B), and C)

REWRITE OF (D): \( ST \circ R = G \circ ST \)
A) What is a circle on the sphere? ... ?? – in class [ANS: linear eqn in the w’s !]

B) \( ST(w_1 + iw_2, w_3) = z = \frac{w_1 + iw_2}{1 - w_3} \)

\( ST^{-1}(z) = (w_1 + iw_2, w_3) = \frac{1}{|z|^2 + 1}(2z, (|z|^2 - 1)) \)

Write out a circle in the w variables? Now

EXER : see what this eqn for w looks like when written in terms of z. Show it is a circle.

C) Rotations of the sphere map circles to circles. (?) Do you buy this?
... finally: D) If $R(w_1, w_2, w_3) = (w_1, -w_2, -w_3)$, then

$$ST(R((w_1, w_2, w_3)) = \frac{w_1 - iw_2}{1 + w_3}$$

If $z = ST(w_1, w_2, w_3)$ then $z = \frac{w_1 + iw_2}{1 - w_3}$ and

$$G \circ ST((w_1, w_2, w_3) = G(z) = 1/z.$$

But

$$1/z = \frac{1 - w_3}{w_1 + iw_2} = \frac{(1 - w_3)(w_1 - iw_2)}{(w_1 + iw_2)(w_1 - iw_2)} = \frac{(1 - w_3)(w_1 - iw_2)}{w_1^2 + w_2^2}$$

But: $w_1^2 + w_2^2 + w_3^2 = 1$ so $w_1^2 + w_2^2 = 1 - w_3^2 = (1 - w_3)(1 + w_3)$ and

$$1/z = \frac{(1 - w_3)(w_1 - iw_2)}{(1 - w_3)(1 + w_3)} = \frac{w_1 - iw_2}{1 + w_3}$$

proving $ST \circ R = G \circ ST$ as desired.