Spheres.
We begin with the sphere in Euclidean three-space but try to quickly move to the general sphere.

The unit sphere in three space is the set of points a unit distance from the origin:

\[ x^2 + y^2 + z^2 = 1 \]

We use the notation \( S^2 \).

Inspired by astronomy, thinking of the light from distant stars impinging on the eye we identify the sphere with the set of rays (or oriented lines) passing through the origin. Then the distance between two rays is simply their angles! It is a number between 0 and \( \pi \).

If we prefer to think of the points \( u, v \in S^2 \) of the sphere as unit vectors \( u, v \) in \( \mathbb{R}^3 \) then

\[ d(u, v) = \cos^{-1}(u \cdot v). \]

is the spherical distance between these points, since the cosine of the angle between \( u \) and \( v \) is \( u \cdot v \).

Verify that this distance formula makes sense for the n-sphere, where

**Definition 1.** The n-sphere \( S^n \) is the locus of points \( \|u\| = 1 \) insider \( \mathbb{R}^{n+1} \) endowed with its standard inner product.

Chord vs intrinsic distance.

Given any subset \( S \subset \mathbb{R}^3 \) we can simply restrict the Euclidean distance \( \|A - B\| = d(A, B) \) to \( A, B \in S \) and thereby get a distance function on \( S \). On the sphere we call this the “chord distance” since it represents the distance of the chord between the two points. This chord distance is NOT the spherical distance defined above.

Instead, we only allow ourselves to connect \( u, v \in S^2 \) by paths which lie entirely on \( S^2 \): that is paths \( x(t) \) in \( \mathbb{R}^3 \) with \( |x(t)| = 1 \). We compute the lengths of those paths as per usual in vector calc: \( \text{length}(x) = \int_a^b |dx/dt| dt \) if \( x : [a, b] \to S^2 \). Then

**Theorem 1.** The spherical distance \( d(u, v) \) is the length of the shortest path on \( S^2 \) which connects \( u \) to \( v \). This shortest path is an arc of a great circle.

**EXERCISE.** Find a formula for the chord distance as a function of \( u \cdot v \).

**OBSERVATION.** Every isometry of \( \mathbb{R}^{n+1} \) which fixes the origin is an isometry of \( S^n \subset \mathbb{R}^{n+1} \).

Indeed, these isometries of \( \mathbb{R}^{n+1} \) which fix the origin are in the standard form \( X \mapsto AX + 0 = AX \) where \( A \) is an \( n+1 \) by \( n+1 \) orthogonal matrix. Now any orthogonal matrix satisfies \( u \cdot v = Au \cdot Av \). Hence \( A \) preserves the spherical distance.

SAS holds on the sphere.

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Spherical area and angle sum formula... see ‘lunes’ in the Reader.

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Homogeneity. Isotropinced.
SAS holding. (or not)
Increasing dimension.....