Rotating from N to P

Finding coordinates of P:

First of all we must find the coordinates of P. Notice that since P has equal x, y and z coordinates, then we can write $P = (t, t, t)$ for some number $t$ which must be positive by the condition that $x > 0$. P is in the unit sphere, which means that it must satisfy $x^2 + y^2 + z^2 = 1$, but for P this equation reduces to:

$$t^2 + t^2 + t^2 = 1 \Rightarrow t = 1/\sqrt{3} \quad (1)$$

So we now know

$$P = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \quad (2)$$

Explaining rotations of $\mathbb{R}^3$:

Every rotation of $\mathbb{R}^3$ can be described by the following two pieces of data:

- A line: the axis of rotation
- An angle: the angle of rotation

Given an axis of rotation $\ell$ and an angle of rotation $\theta$, our rotation won’t move any points in the line $\ell$ and will act as a rotation by an angle $\theta$ on any plane perpendicular to $\ell$. (There’s a small ambiguity here, since we still have to choose which direction to rotate but let’s not worry about that).

This means that to finish this problem we must find the axis of rotation and the angle of the rotation we’re looking for.
Finding the axis of rotation:

We choose an axis of rotation that is perpendicular to the plane through the origin containing both $N$ and $P$. This will be the line in the direction of $N \times P$ (make sure you understand why that’s the case).

We compute $N \times P = \left(-1/\sqrt{3}, 1/\sqrt{3}, 0\right)$

$$\ell = \left\{t(N \times P) \mid t \in \mathbb{R}\right\}$$

(3)

Notice that $\ell$ lies in the $xy$-plane and that by our choice, both $N$ and $P$ are points on the plane through the origin perpendicular to $\ell$. All we have left is to find the correct angle to take $N$ to $P$.

Finding the angle

The angle will simply be the angle between $N$ and $P$ (as vectors based at the origin). The formula is:

$$\theta = \cos^{-1}\left(\frac{N \cdot P}{|N||P|}\right) = \cos^{-1}(N \cdot P) = \cos^{-1}(1/\sqrt{3})$$

(4)
Starting to find the rotation matrix on the standard basis:

Let $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be our rotation. Because of the magic of mathematics it turns out that every rotation is in fact a **linear transformation** this means that we are able to represent our rotation by a matrix once we fix a basis of $\mathbb{R}^3$.

We choose the standard basis of $\mathbb{R}^3$:

\[
e_1 = (1, 0, 0) \\
e_2 = (0, 1, 0) \\
e_3 = (0, 0, 1)
\]

(5)

with this basis the matrix $R$ representing the rotation $\rho$ is

\[
R = \begin{pmatrix}
\uparrow & \uparrow & \uparrow \\
\rho(e_1) & \rho(e_2) & \rho(e_3) \\
\downarrow & \downarrow & \downarrow
\end{pmatrix}
\]

(6)

So that the first column of the matrix $R$ are the coordinates of the vector $\rho(e_1)$ standing up, where the first coordinate is on the top and the third is at the bottom, etc...

Notice that the third column is really easy to compute! That’s because $e_3 = N$, so we have $\rho(e_3) = \rho(N) = P$ simply by our construction of $\rho$. So now we have:

\[
R = \begin{pmatrix}
\uparrow & \uparrow & 1/\sqrt{3} \\
\rho(e_1) & \rho(e_2) & 1/\sqrt{3} \\
\downarrow & \downarrow & 1/\sqrt{3}
\end{pmatrix}
\]

(7)

Our problem now is to compute these column vectors $\rho(e_1)$ and $\rho(e_2)$, but that’s not so immediate, we need a little bit of linear algebra for that.

The problem here is that we know exactly what our rotation does to points on our axis of rotation (the rotation doesn’t move them) and on the plane through the origin perpendicular to the axis (we rotate them by an angle theta), but a point that doesn’t lie on $\ell$ or on its perpendicular plane moves in a weird way that is hard to describe, and the vectors $e_1$ and $e_2$ don’t lie on either of those places.

To solve this problem we define the following vectors

\[
D = (-1/\sqrt{2}, 1/\sqrt{2}, 0) \\
M = (1/\sqrt{2}, 1\sqrt{2}, 0)
\]

(8)
We choose these vectors so that it is easy to describe their image under the rotation $\rho$ (the $\sqrt{2}$’s are only there to make them unit vectors). Since $D \in \ell$ we know $\rho(D) = D$ and since $M$ is in the plane perpendicular to $\ell$ it gets rotated by an angle $\theta$ so that we have:

$$\rho(M) = \cos(\theta) M - \sin(\theta) N$$

(9)

Figure 2: Computing $\rho(M)$

Finishing the computation of matrix $R$:

Now we’re able to compute $\rho(e_1)$ and $\rho(e_2)$, since we know:

$$e_1 = \frac{M}{\sqrt{2}} - \frac{D}{\sqrt{2}}$$

$$e_2 = \frac{M}{\sqrt{2}} + \frac{D}{\sqrt{2}}$$

(10)
We can compute:

\[ \rho(e_1) = \rho \left( \frac{M}{\sqrt{2}} - \frac{D}{\sqrt{2}} \right) \]
\[ = \frac{\rho(M)}{\sqrt{2}} - \frac{\rho(D)}{\sqrt{2}} \]
\[ = \frac{\cos \theta}{\sqrt{2}} M - \frac{\sin \theta}{\sqrt{2}} N - \frac{1}{\sqrt{2}} D \]
\[ = \left( \frac{\cos \theta + 1}{2}, \frac{\cos \theta - 1}{2}, -\frac{\sin \theta}{\sqrt{2}} \right) \]  
\[ \text{(11)} \]

And similarly:

\[ \rho(e_2) = \rho \left( \frac{M}{\sqrt{2}} + \frac{D}{\sqrt{2}} \right) \]
\[ = \frac{\rho(M)}{\sqrt{2}} + \frac{\rho(D)}{\sqrt{2}} \]
\[ = \frac{\cos \theta}{\sqrt{2}} M - \frac{\sin \theta}{\sqrt{2}} N + \frac{1}{\sqrt{2}} D \]
\[ = \left( \frac{\cos \theta - 1}{2}, \frac{\cos \theta + 1}{2}, -\frac{\sin \theta}{\sqrt{2}} \right) \]  
\[ \text{(12)} \]

And finally we obtain the matrix for \( \rho \):

\[ R = \begin{pmatrix}
\frac{\cos \theta + 1}{2} & \frac{\cos \theta - 1}{2} & \frac{1}{\sqrt{3}} \\
\frac{\cos \theta + 1}{2} & \frac{\cos \theta - 1}{2} & \frac{1}{\sqrt{3}} \\
-\frac{\sin \theta}{\sqrt{2}} & -\frac{\sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{3}}
\end{pmatrix} \]  
\[ \text{(13)} \]

And by noticing that \( \cos \theta = 1/\sqrt{3} \) and \( \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{2/3} \) we can write:

\[ R = \begin{pmatrix}
\frac{1+\sqrt{3}}{2\sqrt{3}} & \frac{1-\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1-\sqrt{3}}{2\sqrt{3}} & \frac{1+\sqrt{3}}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix} \]  
\[ \text{(14)} \]